Symmetric Systems and their Applications to Root Systems Extended by Abelian Groups*

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Abstract

We investigate the class of root systems R obtained by extending an irreducible root system by a torsion-free group G. In this context there is a Weyl group $\mathcal{W}$ and a group $\mathcal{U}$ with the presentation by conjugation. We show under additional hypotheses that the kernel of the natural homomorphism $\mathcal{U} \to \mathcal{W}$ is isomorphic to the kernel of $\mathcal{U}^{\mathrm{ab}} \to \mathcal{W}^{\mathrm{ab}}$, where $\mathcal{U}^{\mathrm{ab}}$ and $\mathcal{W}^{\mathrm{ab}}$ denote the abelianizations of $\mathcal{U}$ and $\mathcal{W}$ respectively. For this we introduce the concept of a symmetric system, a discrete version of the concept of a symmetric space.

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Contents

1

1   Introduction

Extended affine root systems (EARSs) belong to the theory of extended affine Lie algebras (EALAs). The class of EALAs is a generalization of the class of affine Kac-Moody algebras. In [AAB$^+$97] EALAs are introduced axiomatically and the EARSs are classified. Lie algebras in this class were studied by physicists under the name of irreducible quasi-simple Lie algebras in [HKT90]. Several years earlier, a certain subclass of EALAs was already studied in [Sai85]. In [Yos04] the notion of a root system extended by an abelian group G is introduced. This more modern approach generalizes the notions of affine root systems given in [AAB$^+$97] and [Sai85] and does not rely on the idea of the root system being embedded discretely in a real vector space. Arguments using discreteness are replaced by algebraic ones and the abelian group G is no longer necessarily free and finitely generated.

In the context of EARSs a group $\mathcal{U}$ is given by the so-called presentation by conjugation:

$$\mathcal{U} \cong \big\langle (\hat{r}_\alpha)_{\alpha \in R^\times} \mid \hat{r}_\alpha = \hat{r}_\beta \quad \text{if } \alpha \text{ and } \beta \text{ are linearly dependent,}$$
$$\hat{r}_\alpha^2 = 1, \ \ \hat{r}_\alpha \hat{r}_\beta \hat{r}_\alpha^{-1} = \hat{r}_{r_\alpha(\beta)}; \ \text{for } \alpha, \beta \in R^\times \big\rangle,$$

where $R^\times$ stands for the set of anisotropic roots. There is a natural group homomorphism from $\mathcal{U}$ into the Weyl group $\mathcal{W}$. Several results are known about whether this homomorphism is injective. (See for example [Kry00], [Aza99], [Aza00], [AS07], [AS06], [Hof07].)

This article attempts to modernize the theory of presentation by conjugation of Weyl groups by extending the class of root systems under consideration to the class proposed by [Yos04] and by choosing a systematic approach using the new notion of a symmetric system. This is a discrete version of the notion of a symmetric space treated in [Loo69]. With our approach the group $\mathcal{U}$ is the initial element of a certain category of reflection groups. We generalize the notion of the Weyl group $\mathcal{W}$ to the case where G is torsion-free and R is tame, that is, G possesses a so-called twist decomposition if R is of non-simply laced type. (This decomposition is a generalization of a decomposition that exists, if G is finitely generated and free abelian.) We obtain the following characterization of the relationship between the group $\mathcal{U}$ and the Weyl group $\mathcal{W}$.

Suppose R is a tame irreducible reduced root system extended by a free abelian group G. Denote the abelianizations of $\mathcal{U}$ and $\mathcal{W}$ by $\mathcal{U}^{\mathrm{ab}}$ and $\mathcal{W}^{\mathrm{ab}}$, respectively.

Theorem   The natural homomorphism $\varphi : \ker(\mathcal{U} \to \mathcal{V}) \to \ker(\mathcal{U}^{\mathrm{ab}} \to \mathcal{W}^{\mathrm{ab}})$ is an isomorphism.

This is the main result of this article and it is proved in the last section. We explore the notion of a symmetric system in Sections 2 to 5 only as far as needed to prove this result. In Section 6 we obtain the necessary results about finite irreducible root systems and the action of their Weyl group on the root lattice and the coroot lattice.

Among other corollaries we derive the following result from the above theorem by investigating the orbits of $\mathcal{W}$ in the root system: Suppose R is a tame irreducible reduced root system extended by a free abelian group G.

Corollary If R is not of type A_1, $B_\ell(\ell \geq 2)$ and $C_\ell(\ell \geq 3)$, then $\mathcal{U} \to \mathcal{W}$ is injective.

In other words, in this case, the Weyl group has the presentation by conjugation. This is a generalization of results known in the case where G is a finitely generated free abelian group. We expect that various other results will be generalized using the theorem above and by investigating the groups $\mathcal{U}^{\mathrm{ab}}$ and $\mathcal{W}^{\mathrm{ab}}$, which are elementary abelian two-groups and thus are accessible to arguments of linear algebra over the Galois field GF(2).

2   Symmetric systems

In this section we introduce the notion of symmetric system. Its definition requires two of the axioms of a symmetric space in the sense of [Loo69]. To a given symmetric system T we associate the category of T-reflection groups. It is an important observation that every morphism from one reflection group to another constitutes a central extension. The category turns out to be very well-behaved: Morphisms are uniquely determined by the reflection groups from which and into which they are defined. The category can be understood as a complete lattice.

Definition 2.1 (Symmetric system) Let T be a set with a (not necessarily associative) multiplication

$$\mu : T \times T \to T, \ (s,t) \mapsto s.t.$$

Then the pair (T, μ) is called a *symmetric system* if the following conditions are satisfied for all s, t and $r \in T$:

(S1) $s.(s.t) = t,$

(S2) $r.(s.t) = (r.s).(r.t).$

By abuse of language, we will sometimes say that T is a symmetric system instead of saying that (T, μ) is a symmetric system. If $s.t = t$ for all s and $t \in T$ then we call μ the *trivial multiplication*. If s and $t \in T$ we write $s \perp t$ if $s \neq t$, $s.t = t$ and $t.s = s$.   ◇

Example 2.2 Let $\mathcal{X}$ be a group with a subset T of involutions that is invariant under conjugation. Then the multiplication

$$T \times T \to T, \ (t,s) \mapsto t.s = tst^{-1}$$

turns T into a symmetric system. We have $s \perp t$ if and only if s and t are distinct and commute.

For the remainder of this section, let T be a symmetric system.

Definition 2.3 (Reflection group) Let $\mathfrak{X}$ be a group acting on T. We will denote the element in T obtained by x acting on t by $x.t$. Let

$$\bullet^{\mathfrak{X}} : \ T \to \mathfrak{X}, \ t \mapsto t^{\mathfrak{X}}$$

be a function. Then $(\mathfrak{X}, \bullet^{\mathfrak{X}})$ is called a *T-reflection group*, if the following conditions are satisfied:

(G1) The group $\mathfrak{X}$ is generated by the set $T^{\mathfrak{X}} := \{ t^{\mathfrak{X}} \mid t \in T \}$.

(G2) For all s and $t \in T$ we have $t^{\mathfrak{X}}.s = t.s$.

(G3) For all s and $t \in T$ we have $t^{\mathfrak{X}}.s^{\mathfrak{X}} \ = \ (t.s)^{\mathfrak{X}}$.

(G4) For every $t \in T$ we have $(t^{\mathfrak{X}})^2 = 1$.

If we do not need to specify the map $\bullet^{\mathfrak{X}}$ we will also say that $\mathfrak{X}$ is a reflection group instead of saying that $(\mathfrak{X}, \bullet^{\mathfrak{X}})$ is a reflection group. We say that a reflection group $(\mathfrak{X}, \bullet^{\mathfrak{X}})$ *separates reflections* if the map $\bullet^{\mathfrak{X}}$ is injective and that $\mathfrak{X}$ is *proper* if it separates reflections and $t^{\mathfrak{X}} \neq \mathrm{id}$ for every $t \in T$. We say that T itself is *proper* if it has a proper reflection group. $\diamond$

Example 2.4 In the context of Example 2.2, the group $\mathfrak{X}$ can be viewed as a proper reflection group for T, if it is generated by $T^{\mathfrak{X}}$.

Remark 2.5 If T is proper, then we immediately obtain $s.s = s$ and $s.t = t \iff t.s = s$ for all s and $t \in T$ by considering the images in the proper reflection group. In that case T is a discrete symmetric space in the sense of [Loo69]. $\diamond$

Remark 2.6 If $\mathfrak{X}$ is a T-reflection group, then we have

$$x.(t^{\mathfrak{X}}) = (x.t)^{\mathfrak{X}}$$

for all $x \in \mathfrak{X}$ and $t \in T$, since $T^{\mathfrak{X}}$ generates $\mathfrak{X}$. If $\mathfrak{Y}$ is another T-reflection group and if $s_1, \ldots, s_n$ and $t \in T$ then

$$(s_1^{\mathfrak{X}} \cdots s_n^{\mathfrak{X}}).t = (s_1^{\mathfrak{Y}} \cdots s_n^{\mathfrak{Y}}).t.$$

This means that the $\mathfrak{X}$-orbits and the $\mathfrak{Y}$-orbits in T are the same. $\diamond$

Definition 2.7 (Reflection morphism) Let T be a symmetric system and let $\mathfrak{X}$ and $\mathfrak{Y}$ be T-reflection groups. Then a group homomorphism $\mathfrak{X} \to \mathfrak{Y}$ is called a *T-reflection morphism* or a *T-morphism*, if the following diagram is commutative:

$\diamond$

If reflection groups $\mathcal{X}$ and $\mathcal{Y}$ are given, then there is at most one morphism $\mathcal{X} \to \mathcal{Y}$, since the images of a generating set are prescribed. Every reflection morphism is surjective, since it has a generating set in its image. If $\zeta : \mathcal{X} \to \mathcal{Y}$ is a reflection morphism, then the actions of $\mathcal{X}$ and $\mathcal{Y}$ on T are compatible in the following sense: We have $x.t = \zeta(x).t.$ for all $x \in \mathcal{X}$ and $t \in T$.

Definition 2.8 (Category of T-reflection groups) Let T be a symmetric system. Then we associate to it the *category of T-reflection groups* as follows: The objects of the category are the T-isomorphy classes of reflection groups. The morphisms in this category are given by the reflection morphisms. ◇

Lemma 2.9 *Suppose $\mathcal{X}$ and $\mathcal{Z}$ are T-reflection groups. Let $\mathcal{Y}$ be a group and suppose there are group homomorphisms*

$$\mathcal{X} \xrightarrow{\varphi} \mathcal{Y} \xrightarrow{\psi} \mathcal{Z},$$

such that φ is surjective and the composition $\psi \circ \varphi$ is a T-reflection morphism. Then the map

$$\bullet^{\mathcal{Y}} : \ T \to \mathcal{Y}, \ t \mapsto \varphi(t^{\mathcal{X}})$$

turns the pair $(\mathcal{Y}, \bullet^{\mathcal{Y}})$ into a T-reflection group.

Proof. The homomorphism ψ induces an action of $\mathcal{Y}$ on T. We will verify the axioms of Definition 2.3. Since φ is surjective, we have (G1). If s and $t \in T$ then

$$t^{\mathcal{Y}}.s \ = \ \varphi(t^{\mathcal{X}}).s \ = \ \psi \circ \varphi(t^{\mathcal{X}}).s \ = \ t^{\mathcal{Z}}.s \ = \ t.s,$$

so (G2) is satisfied. Finally, we turn to (G3) and (G4). For all $s, t \in T$ we have

$$(t^{\mathcal{Y}})^2 = \left(\varphi(t^{\mathcal{X}})\right)^2 = \varphi\left((t^{\mathcal{X}})^2\right) \ = \ 1 \quad \text{and}$$
$$t^{\mathcal{Y}}.s^{\mathcal{Y}} = t^{\mathcal{Y}}s^{\mathcal{Y}}(t^{\mathcal{Y}})^{-1} = \varphi(t^{\mathcal{X}}s^{\mathcal{X}}(t^{\mathcal{X}})^{-1}) = \varphi(t^{\mathcal{X}}.s^{\mathcal{X}}) = \varphi\left((t.s)^{\mathcal{X}}\right) = (t.s)^{\mathcal{Y}} \qquad ■$$

Though quickly proved, the following Lemma is an important observation about T-morphisms.

Lemma 2.10 *Every T-morphism $\zeta : \ \mathcal{X} \to \mathcal{Y}$ is a central extension.*

Proof. Let $x \in \ker \zeta$. Then we have

$$xs^{\mathcal{X}}x^{-1} = x.s^{\mathcal{X}} = (x.s)^{\mathcal{X}} = (\zeta(x).s)^{\mathcal{X}} = s^{\mathcal{X}}.$$

for every $s \in T$. This proves the claim. ■

Definition 2.11 (Initial and terminal reflection group) A T-reflection group $\mathcal{U}$ is called *initial* if the following universal property is satisfied: For every T-reflection group $\mathcal{X}$ there is a unique reflection morphism $\mathcal{U} \to \mathcal{X}$.

A T-reflection group $\mathcal{T}$ is called *terminal* if the following universal property is satisfied: For every T-reflection group $\mathcal{X}$ there is a unique reflection morphism $\mathcal{X} \to \mathcal{T}$. ◇

Proposition 2.12 *An initial and a terminal T-reflection group exist.*

Proof. For $s \in T$ set
$$s^{\mathcal{T}} : \ T \to T, \ t \mapsto s.t$$
and note that $s^{\mathcal{T}}$ is its own inverse and thus a bijection due to axiom (S1). Let $\mathcal{T}$ be the subgroup of the symmetric group over T generated by $T^{\mathcal{T}} = \{s^{\mathcal{T}} \mid s \in T\}$. We will show that the pair $(\mathcal{T}, \cdot^{\mathcal{T}})$ is a reflection group. There is a natural action of $\mathcal{T}$ on T. Axiom (G1) is satisfied by definition and if s and $t \in T$ then $t^{\mathcal{T}}.s = t.s$ by definition, so axiom (G2) is satisfied. Now let r, s and $t \in T$. Then

$$(t^{\mathcal{T}})^2.r = t^{\mathcal{T}}.(t^{\mathcal{T}}.r) = t.(t.r) = r \quad \text{by (S1)} \quad \text{and}$$
$$(t^{\mathcal{T}}.s^{\mathcal{T}}).r = \left(t^{\mathcal{T}} s^{\mathcal{T}} (t^{\mathcal{T}})^{-1}\right).r = t.\left(s.(t.r)\right) = (t.s).\left(t.(t.r)\right) \quad \text{by (S2)}$$
$$= (t.s).r = (t.s)^{\mathcal{T}}.r,$$

so axioms (G3) and (G4) are satisfied. Now we show that $\mathcal{T}$ is terminal. If $\mathcal{X}$ is another reflection group then the action of $\mathcal{X}$ on T induces a group homomorphism into the symmetric group on T. Its image is in $\mathcal{T}$ by (G2). This group homomorphism is a reflection morphism.

Now consider the group $\mathcal{U}$ with the presentation

$$\mathcal{U} := \left\langle (t^{\mathcal{U}})_{t \in T} \mid (t^{\mathcal{U}})^2 = 1 \ \text{ and } \ t^{\mathcal{U}} s^{\mathcal{U}} (t^{\mathcal{U}})^{-1} = (t.s)^{\mathcal{U}} \ \text{ for } s \text{ and } t \in T \right\rangle \quad (1)$$

with the natural map $\cdot^{\mathcal{U}} : \ T \to \mathcal{U}, \ t \mapsto t^{\mathcal{U}}$. Let $\mathcal{X}$ be a reflection group. Since $(t^{\mathcal{X}})^2 = 1$ and $t^{\mathcal{X}}.s^{\mathcal{X}} = (t.s)^{\mathcal{X}}$ for all s and $t \in T$, there is a group homomorphism $\varphi : \ \mathcal{U} \to \mathcal{X}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
 & T & \\
\swarrow & & \searrow \\
\mathcal{U} & \xrightarrow{\ \ \varphi\ \ } & \mathcal{X}.
\end{array}
$$

This is true in particular for $\mathcal{X} := \mathcal{T}$, which induces an action of $\mathcal{U}$ on T. In this way, $\mathcal{U}$ becomes a reflection group, and it is initial. ∎

Example 2.13 If $(\mathcal{W}, S)$ is a Coxeter system, then $\mathcal{W}$ is the initial reflection group for the symmetric system $T := W.S := \{wsw^{-1} \mid s \in S, \ w \in \mathcal{W}\}$ in the sense of Example 2.2. (See [Hof07] Proposition 4.2, for instance.)

Lemma 2.14 *If $\mathcal{X}$ is a T-reflection group that separates reflections, then its center is the kernel of the reflection morphism $\mathcal{X} \to \mathcal{T}$, where $\mathcal{T}$ is the terminal T-reflection group.*

Proof. Let $\mathcal{X}$ be a T-reflection group that separates reflections. Denote its center by $\mathcal{Z}$. Let $\mathcal{Y} = \mathcal{X}/\mathcal{Z}$ and denote by $\zeta : \ \mathcal{X} \to \mathcal{Y}$ the quotient morphism. Define $\cdot^{\mathcal{Y}} : \ T \to \mathcal{Y}$ as the composition of $\cdot^{\mathcal{X}}$ with ζ. Now we have $(z.t)^{\mathcal{X}} =$

$z.t^{\mathcal{X}} = t^{\mathcal{X}}$ for every $z \in \mathcal{Z}$. Since $\mathcal{X}$ separates reflections, we conclude that $\mathcal{Z}$ acts trivially on T. So there is an action of $\mathcal{Y}$ on T satisfying $\zeta(x).t = x.t$ for all $x \in \mathcal{X}$ and $t \in T$. This action turns $(\mathcal{Y}, \cdot^{\mathcal{Y}})$ into a reflection group and ζ into a reflection morphism. Since $\mathcal{X} \to \mathcal{T}$ is a central extension, there is a reflection morphism $\mathcal{T} \to \mathcal{Y}$. Since $\mathcal{T}$ is terminal, it is an isomorphism. This entails $\mathcal{Z} = \ker(\mathcal{X} \to \mathcal{T})$. ∎

Denote the initial T-reflection group by $\mathcal{U}$ and the terminal reflection group by $\mathcal{T}$. Denote the kernel of $\mathcal{U} \to \mathcal{T}$ by $\mathcal{A}$. Let $\mathcal{Z}$ be a subgroup of $\mathcal{A}$. Lemmas 2.9 and 2.10 taken together yield:

Theorem 2.15 *There is one-to-one correspondence between subgroups of the abelian group $\mathcal{A}$ and isomorphy classes of T-reflection groups. It is given by*

$$\mathcal{Z} \mapsto \mathcal{U}/\mathcal{Z}.$$

There is a reflection morphism $\mathcal{U}/\mathcal{Z}_1 \to \mathcal{U}/\mathcal{Z}_2$ if and only if $\mathcal{Z}_1 \subseteq \mathcal{Z}_2$.

In other words, the category of T-reflection groups is small and is isomorphic to the complete lattice of subgroups of $\mathcal{A}$ viewed as a category in the usual way.

3 Functors on categories of reflection groups

In this section we introduce the notion of a system morphism from one symmetric system to another. Such a morphism gives rise to a functor between the corresponding categories. An important example of such a functor is the so-called abelianization functor that comes from a very natural system morphism that can be defined on any symmetric system. The abelianization functor maps a reflection group to its abelianization.

Suppose (T, ι_T) and (S, ι_S) are symmetric systems.

Definition 3.1 (System morphism) A map $\overline{\cdot}: T \to S,\ t \mapsto \overline{t}$ is called a *symmetric system morphism* or a *system morphism*, if $\overline{t.s} = \overline{t}.\overline{s}$ for all s and $t \in T$. ◇

Definition 3.2 Let $\overline{\cdot}: T \to S$, be a system morphism. Let $\mathcal{X}$ be a T-reflection group and let $\mathcal{P}$ be an S-reflection group. A group homomorphism $\varphi: \mathcal{X} \to \mathcal{P}$ is called $\overline{\cdot}$-*compatible* if $\varphi(t^{\mathcal{X}}) = \overline{t}^{\mathcal{P}}$ for every $t \in T$, i.e. the following diagram commutes:

$$
\begin{array}{ccc}
T & \longrightarrow & S \\
\downarrow & & \downarrow \\
\mathcal{X} & \xrightarrow{\ \varphi\ } & \mathcal{P}.
\end{array}
$$
◇

Lemma 3.3 *Let $\mathcal{U}$ be the initial T-reflection group, and let $\mathcal{V}$ be the initial S-reflection group. Then there is a $\overline{\cdot}$-compatible group homomorphism $\mathcal{U} \to \mathcal{V}$.*

Proof. It suffices to show that the assignment

$$\varphi: \ t^{\mathcal{U}} \mapsto \overline{t}^{\mathcal{V}},$$

for every $t \in T$, can be extended to a group homomorphism $\mathcal{U} \to \mathcal{V}$. This follows from the fact that we have

$$\left(\varphi(t^{\mathcal{U}})\right)^2 = (\overline{t}^{\mathcal{V}})^2 \ = \ 1 \quad \text{and}$$

$$\varphi(t^{\mathcal{U}})\varphi(s^{\mathcal{U}})\left(\varphi(t^{\mathcal{U}})\right)^{-1}\left(\varphi((t.s)^{\mathcal{U}})\right)^{-1} = \overline{t}^{\mathcal{V}}\overline{s}^{\mathcal{V}}(\overline{t}^{\mathcal{V}})^{-1}(\overline{t.s}^{\mathcal{V}})^{-1}$$

$$= \overline{t}^{\mathcal{V}}\overline{s}^{\mathcal{V}}(\overline{t}^{\mathcal{V}})^{-1}((\overline{t}.\overline{s})^{\mathcal{V}})^{-1}$$

$$= 1$$

for all s and $t \in T$. ∎

Corollary 3.4 *If $\mathcal{U}$ is the initial T-reflection group and $\mathcal{P}$ is any S-reflection group, then there is a $\overline{\cdot}$-compatible $\mathcal{U} \to \mathcal{P}$.*

Proof. Let $\mathcal{V}$ be the initial S-reflection group. Then this follows from diagram chasing in

∎

Lemma 3.5 *Let $\mathcal{X}$ be the terminal T-reflection group, and let $\mathcal{P}$ be the terminal S-reflection group. Then there is a $\overline{\cdot}$-compatible group homomorphism $\mathcal{X} \to \mathcal{P}$.*

Proof. If t and $r \in T$ then

$$\overline{t^{\mathcal{X}}.r} = \overline{t.r} = \overline{t}.\overline{r} = \overline{t}^{\mathcal{P}}.\overline{r},$$

in other words, the following diagram commutes:

In view of how terminal reflection groups are constructed in the proof of Proposition 2.12, this means that the assignment

$$T^{\mathcal{X}} \to \mathcal{P}, \ t^{\mathcal{X}} \mapsto \overline{t}^{\mathcal{P}}$$

can be continued to a group homomorphism $\mathcal{X} \to \mathcal{P}$. It is compatible. ∎

Corollary 3.6 *If $\mathcal{P}$ is the terminal S-reflection group and $\mathcal{X}$ is any T-reflection group, then there is a $\overline{\cdot}$-compatible $\mathcal{X} \to \mathcal{P}$.*

Let $\mathcal{X}$ be a T reflection group. Let M be the set of all S-reflection groups $(\mathcal{P}, \cdot^{\mathcal{P}})$ modulo equivalence such that there is a compatible $\varphi_{\mathcal{P}} : \mathcal{X} \to \mathcal{P}$. (Due to Theorem 2.15, this is a set.) This set is not empty by Corollary 3.6. Let $\overline{\mathcal{X}}$ be the subgroup of the group

$$\prod_{\mathcal{P} \in \mathbf{M}} \mathcal{P}$$

generated by the elements

$$s^{\overline{\mathcal{X}}} = \prod_{\mathcal{P} \in M} s^{\mathcal{P}}$$

with $s \in S$. Then $\overline{\mathcal{X}}, \cdot^{\mathcal{X}}$ is an S-reflection group. It is actually the limit of the groups in M. The group homomorphism

$$\varphi : \mathcal{X} \to \overline{\mathcal{X}}, \ x \mapsto \prod_{\mathcal{P} \in M} \varphi_{\mathcal{P}}(x)$$

is compatible.

If $\zeta : \mathcal{X} \to \mathcal{Y}$ is a T-reflection morphism then we denote by $\overline{\zeta}$ the unique S-reflection morphism from $\overline{\mathcal{X}}$ to $\overline{\mathcal{Y}}$. We have obtained

Proposition 3.7 *If $\overline{\cdot} : T \to S$ is a system morphism, then the assignments $\mathcal{X} \mapsto \overline{\mathcal{X}}$ and $\zeta \mapsto \overline{\zeta}$ provide a functor from the category of T-reflection groups to the category of S-reflection groups and there is a $\overline{\cdot}$-compatible group homomorphism $\mathcal{X} \to \overline{\mathcal{X}}$.*

Remark 3.8 By Lemma 3.3, the functor maps initial reflection groups to initial reflection groups.   ◇

Lemma 3.9 *If $\overline{\cdot} : T \to S$ is a surjective system morphism, then the kernel of $\mathcal{X} \to \overline{\mathcal{X}}$ is the subgroup of $\mathcal{X}$ generated by*

$$M := \{ s^{\mathcal{X}} t^{\mathcal{X}} \mid s, t \in T \ with \ \overline{s} = \overline{t} \}.$$

Proof. Set $\mathcal{K} := \ker(\mathcal{X} \to \overline{\mathcal{X}})$ and let $\mathcal{N}$ be the subgroup of $\mathcal{X}$ generated by M. Since $M \subseteq \mathcal{K}$ we have $\mathcal{N} \subseteq \mathcal{K}$. Note that M is invariant under conjugation by elements in $\mathcal{X}$, so $\mathcal{N}$ is a normal subgroup. Let $\mathcal{P} := \mathcal{X}/\mathcal{N}$ and let $\varphi : \mathcal{X} \to \mathcal{P}$ be the quotient morphism. Since $\varphi(s^{\mathcal{X}}) = \varphi(t^{\mathcal{X}})$ for s and $t \in T$ with $\overline{s} = \overline{t}$ and since $\overline{\cdot}$ is surjective the assignment $\overline{t} \mapsto \varphi(t^{\mathcal{X}})$ provides a function $\cdot^{\mathcal{P}} : S \to \mathcal{P}$.

Due to Corollary 3.6 there is an action of $\mathcal{X}$ on S. Since $\mathcal{N}$ acts trivially on S, we can define an action of $\mathcal{P}$ on S via $\varphi(x).s = x.s$ for $x \in \mathcal{X}$ and $s \in S$. It is a standard computation to verify that $(\mathcal{P}, \cdot^{\mathcal{P}})$ becomes an S-reflection group with this action and that φ is $\overline{\cdot}$-compatible. So there is an S-reflection morphism $\overline{\mathcal{X}} \to \mathcal{P}$, which means $\mathcal{K} \subseteq \mathcal{N}$.   ∎

Example 3.10 Let T be a symmetric system and let S be a singleton with the trivial multiplication. The only possible map $f : T \to S$ is a system morphism. If $\mathcal{U}$ is the initial T reflection group, then there is a group homomorphism $\det : \mathcal{U} \to \{1, -1\}$ sending the generators $T^{\mathcal{U}}$ to -1. Because of this property, we call the functor discussed in this example the *determinant functor*.

The functor constructed in the following plays an important role in the last section, when our main result is stated.

Construction 3.11 (The abelianization functor) Let T be a symmetric system with terminal reflection group $\mathcal{T}$. We denote the orbit space by T^{ab} and the orbit map by $\bullet^{\mathrm{ab}} : T \to T^{\mathrm{ab}}, \ t \mapsto t^{\mathrm{ab}}$. With the trivial multiplication the set T^{ab} becomes a symmetric system and $\bullet^{\mathrm{ab}}$ is a system morphism due to

$$(s.t)^{\mathrm{ab}} = (s^{\mathcal{T}}.t)^{\mathrm{ab}} = t^{\mathrm{ab}} = s^{\mathrm{ab}}.t^{\mathrm{ab}}$$

for all s and $t \in T$. So, by Proposition 3.7, we have a functor ab from the category of T-reflection groups to the category of T^{ab}-reflection groups.

Let $\mathcal{Y}$ be a T-reflection group. If s and $t \in T$ then the commutator of the generators $s^{\mathcal{Y}}$ and $t^{\mathcal{Y}}$ is given by

$$[s^{\mathcal{Y}}, t^{\mathcal{Y}}] = s^{\mathcal{Y}} t^{\mathcal{Y}} (s^{\mathcal{Y}})^{-1} (t^{\mathcal{Y}})^{-1} = (s^{\mathcal{Y}}.t^{\mathcal{Y}}) t^{\mathcal{Y}} = (s.t)^{\mathcal{Y}} t^{\mathcal{Y}}$$

So, in view of Lemma 3.9 the image $\mathcal{Y}^{\mathrm{ab}}$ of $\mathcal{Y}$ under the *ab*-functor is precisely the abelianization of $\mathcal{Y}$. The initial T^{ab}-reflection group $\mathcal{V}$ has the presentation

$$\langle (s^{\mathcal{V}})_{s \in T^{\mathrm{ab}}} \mid (s^{\mathcal{V}})^2 = 1, \ s^{\mathcal{V}} t^{\mathcal{V}} = t^{\mathcal{V}} s^{\mathcal{V}} \quad \text{for } s \text{ and } t \in T^{\mathrm{ab}} \rangle.$$

So it is isomorphic to $\mathbb{Z}_2^{T^{\mathrm{ab}}}$ and it is a proper reflection group. $\diamond$

Corollary 3.12 *If $\mathcal{U}$ is the initial T-reflection group, then $T^{\mathcal{U}}$ is a minimal $\mathcal{U}$-invariant generating set for it.*

Definition 3.13 (The abelianization functor) The functor discussed in Construction 3.11 is called the *abelianization functor*. $\diamond$

Remark 3.14 If $\bar{\bullet} : T \to S$ is a surjective system morphism, then there is a unique system morphism $T^{\mathrm{ab}} \to S^{\mathrm{ab}}$ such that the following diagram commutes:

$$\begin{array}{ccc} T & \longrightarrow & T^{\mathrm{ab}} \\ \downarrow & & \downarrow \\ S & \longrightarrow & S^{\mathrm{ab}}. \end{array}$$

$\diamond$

Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{P}$ and $\mathcal{Q}$ be groups and suppose we have the commutative diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{P} & \longrightarrow & \mathcal{Q}, \end{array} \tag{2}$$

where all the arrows represent surjective group homomorphisms. We will use the notation

$$\mathcal{K}_{\mathcal{Q}}^{\mathcal{Y}} = \ker(\mathcal{Y} \to \mathcal{Q}) \tag{3}$$

and omit the subscript, if no confusion arises. The following lemma about short exact sequences will be needed later on:

Lemma 3.15 *Diagram (2) can be extended to the following diagram with three vertical and three horizontal short exact sequences:*

Proof. By [Bou98] Ch I § 1.4 Proposition 2 the sequence

$$1 \to \ker(\mathcal{K}_{\mathcal{P}}^{\mathcal{X}} \to \mathcal{K}_{\mathcal{Q}}^{\mathcal{Y}}) \to \ker(\mathcal{X} \to \mathcal{Y}) \to \ker(\mathcal{P} \to \mathcal{Q}) \to 1$$

is part of a longer exact sequence. Although that proposition is stated for abelian groups, the proof works for arbitrary groups, as well. ∎

Let S and T be symmetric systems and let $\bar{\cdot} : T \to S$ be a system morphism.

Definition 3.16 (System section) A system morphism $\dot{\cdot} : S \to T$ is called a *system section (for $\bar{\cdot}$)*, if it is a right inverse of $\bar{\cdot}$, (i.e. $\overline{\dot{s}} = s$ for all $s \in S$). If such a system section is given and $t \in T$, we will sometimes write $\dot{t}$ instead of $\dot{\bar{t}}$ to keep notation simple. ◇

Remark 3.17 Suppose $\dot{\cdot} : S \to T$ is a system section for $\bar{\cdot} : T \to S$. Let $\mathcal{U}$ be the initial T-reflection group and let $\mathcal{V}$ be the initial S-reflection group. If there is a $\bar{\cdot}$-compatible group homomorphism $\mathcal{X} \to \mathcal{V}$, then the diagram

shows that there is a homomorphic section $\mathcal{V} \to \mathcal{X}$ for $\mathcal{X} \to \mathcal{V}$. With the notation established in (3) this means $\mathcal{X} \cong \mathcal{K}_{\mathcal{V}}^{\mathcal{X}} \rtimes \mathcal{V}$. We will set

$$t^{\mathcal{K}_{\mathcal{V}}^{\mathcal{X}}} := t^{\mathcal{X}} \dot{\imath}^{\mathcal{X}} \in \mathcal{K}_{\mathcal{V}}^{\mathcal{X}}.$$

In this way we have $t^{\mathcal{X}} \cong (t^{\mathcal{K}_{\mathcal{V}}^{\mathcal{X}}}, \overline{t}^{\mathcal{V}})$.   ⋄

4   Symmetric systems extended by an abelian group

For the entire section let S and T be symmetric systems and let $\overline{\cdot} : T \to S$ be a system morphism with a system section $\dot{\cdot}$. Denote the initial T-reflection group by $\mathcal{U}$ and the initial S-reflection group by $\mathcal{V}$. Suppose that the terminal T reflection group $\mathcal{A}$ satisfies $\overline{\mathcal{A}} = \mathcal{V}$ and suppose that $\mathcal{K} := \ker(\mathcal{A} \to \mathcal{V})$ is abelian. In other words $\mathcal{A}$ is an extension of the S-reflection group $\mathcal{V}$ by the abelian group $\mathcal{K}$. We will write the group $\mathcal{K}$ additively.

Note that the hypotheses imply that $\overline{\mathcal{X}} = \mathcal{V}$ for any T-reflection group $\mathcal{X}$. The group homomorphism $\mathcal{X} \to \mathcal{V}$ has a section $\mathcal{V} \to \mathcal{A}$ by Remark 3.17. So $\mathcal{X}$ is isomorphic to the semidirect product $\mathcal{K}^{\mathcal{X}} \rtimes \mathcal{V}$.

Let $\mathcal{X}$ be a T-reflection group. Set

$$\mathcal{X}_{\mathrm{ab}} := \mathcal{X}/(\mathcal{K}^{\mathcal{X}})' \quad \text{and} \quad \cdot^{\mathcal{X}_{\mathrm{ab}}} : T \to \mathcal{X}_{\mathrm{ab}}, \ t \mapsto t^{\mathcal{X}_{\mathrm{ab}}} = q(t^{\mathcal{X}}),$$

where $q : \mathcal{X} \to \mathcal{X}_{\mathrm{ab}}$ denotes the quotient map.

Lemma 4.1 *The pair* $(\mathcal{X}_{\mathrm{ab}}, \cdot^{\mathcal{X}_{\mathrm{ab}}})$ *is a T-reflection group.*

Proof. By Lemma 3.15 we have the following commutative diagram:

Since $\mathcal{K}^{\mathcal{A}}$ is abelian, the kernel $\ker \psi$ contains the commutator subgroup $(\mathcal{K}^{\mathcal{X}})'$. So ψ factors through $\mathcal{X}_{\mathrm{ab}}$ yielding group homomorphisms

$$\mathcal{X} \longrightarrow \mathcal{X}_{\mathrm{ab}} \longrightarrow \mathcal{A}.$$

We are done by Lemma 2.9.   ∎

Let $\mathcal{B}$ be a T-reflection group such that $\mathcal{B} = \mathcal{B}_{\mathrm{ab}}$, i.e. such that $\mathcal{K}^{\mathcal{B}} := \mathcal{K}_{\mathcal{V}}^{\mathcal{B}}$ is abelian. We have the following commuting diagram:

Proposition 4.2 *If $\mathcal{K}$ is free abelian, then φ is an isomorphism.*

Proof. The central extension $\mathcal{K}^{\mathcal{B}} \to \mathcal{K}$ splits, since $\mathcal{K}$ is free abelian and $\mathcal{K}^{\mathcal{B}}$ is abelian. So we have $\mathcal{K}^{\mathcal{B}} \cong \mathcal{Z} \times \mathcal{K}$ for some abelian group $\mathcal{Z}$. By Remark 3.17, we have $\mathcal{B} \cong \mathcal{K}^{\mathcal{B}} \rtimes \mathcal{V}$ and $\mathcal{A} \cong \mathcal{K} \rtimes \mathcal{V}$. We can identify $\ker(\mathcal{A} \to \mathcal{B})$ with $\mathcal{Z}$. Since $\mathcal{Z}$ is central in $\mathcal{B}$, the action of $\mathcal{V}$ on it is trivial. This implies

$$\mathcal{B} \cong (Z \times \mathcal{K}) \rtimes \mathcal{V} \cong Z \times (\mathcal{K} \rtimes \mathcal{V}) \cong \mathcal{Z} \times \mathcal{A}.$$

So the diagram above becomes

$$
\begin{array}{ccccc}
\mathcal{Z} & \longrightarrow & \mathcal{Z} \times \mathcal{A} & \longrightarrow & \mathcal{A} \\
\mathrm{id}\downarrow & & \downarrow & & \downarrow \\
\mathcal{Z} & \longrightarrow & \mathcal{Z} \times \mathcal{A}^{\mathrm{ab}} & \longrightarrow & \mathcal{A}^{\mathrm{ab}}
\end{array}
$$

The identity map $\mathrm{id} : \mathcal{Z} \to \mathcal{Z}$ is the unique map that makes this diagram commute. ∎

5 Extensions using bihomomorphisms

We continue to work in the setting of the previous section. In this section we will see how certain bihomomorphisms allow us to construct new reflection groups from $\mathcal{A}$. This process could be investigated more generally using arbitrary cocycles. However the alternating bihomomorphic ones are the ones that we need in order to construct the Weyl group in the last section.

Suppose that $\mathcal{Z}$ is an abelian group and that $c : \mathcal{K}^2 \to \mathcal{Z}$ is an alternating bihomomorphism that is $\mathcal{V}$-invariant in the following sense: If $k, k' \in \mathcal{K}$ and $v \in \mathcal{V}$, then $c(v.k, v.k') = c(k, k')$. Now consider the group $\mathcal{K}^c := \mathcal{Z} \times_c \mathcal{K}$ with the multiplication

$$(\mathcal{K}^c)^2 \to \mathcal{K}^c, \ \big((z_1, k_1), (z_2, k_2) \big) \mapsto (z_1 + z_2 + c(k_1, k_2), k_1 k_2)$$

and note that $\mathcal{V}$ acts on it by group automorphisms via

$$v.(z, k) = (z, v.k)$$

for $v \in \mathcal{V}$, $z \in \mathcal{Z}$ and $k \in \mathcal{K}$. So we can form the semidirect product $\mathcal{K}^c \rtimes \mathcal{V}$. Set

$$\bullet^{\mathcal{X}} : \ T \to \mathcal{K}^c \rtimes \mathcal{V}, \ t \mapsto t^{\mathcal{X}} = (0, t^{\mathcal{K}}, \bar{t}^{\mathcal{V}})$$

and let $\mathcal{X}$ be the subgroup of $\mathcal{K}^c \rtimes \mathcal{V}$ generated by $T^{\mathcal{X}}$. The projection $\mathcal{K}^c \rtimes \mathcal{V} \to \mathcal{K} \rtimes \mathcal{V}$ is a group homomorphism which restricts to one $\mathcal{X} \to \mathcal{A}$ such that $t^{\mathcal{X}}$ is mapped to $t^{\mathcal{A}}$ for every $t \in T$. Via this homomorphism we obtain an action of $\mathcal{X}$ on T.

Definition 5.1 We will denote the pair $(\mathcal{X}, \cdot^{\mathcal{X}})$ by $(\mathcal{A}^c, \cdot^{\mathcal{A}^c})$ and call c a *reflection bihomomorphism (for $\mathcal{A}$)* if $\mathcal{A}^c$ is a T-reflection group. $\diamond$

If c is a reflection bihomomorphism, then $\mathcal{A}^c \to \mathcal{A}$ is a reflection morphism. So $\overline{\mathcal{A}^c} = \mathcal{V}$ and $\mathcal{K}_{\mathcal{V}}^{\mathcal{A}^c}$ is the subgroup of $\mathcal{K}^c$ generated by $\{0\} \times_c T^{\mathcal{K}}$.

Lemma 5.2 *An alternating bihomomorphism $c : \mathcal{K}^2 \to \mathcal{Z}$ is a reflection bihomomorphism if and only if $c(s^{\mathcal{V}}.t^{\mathcal{K}}, t^{\mathcal{K}}) = 0$ for every s and $t \in T$.*

Proof. Set $\mathcal{X} := \mathcal{Y}^c$ and note that

$$\bar{t}^{\mathcal{V}}.t^{\mathcal{K}} = i^{\mathcal{X}} t^{\mathcal{X}} i^{\mathcal{X}} i^{\mathcal{X}} = (t^{\mathcal{K}})^{-1}$$

in $\mathcal{A}$ for every $t \in T$. Written additively in $\mathcal{K}$ this means $\bar{t}^{\mathcal{V}}.t^{\mathcal{K}} = -t^{\mathcal{K}}$. Axioms (G1) and (G2) are satisfied. If s and $t \in T$ then

$$
\begin{aligned}
(t^{\mathcal{X}})^2 &= \left(0, t^{\mathcal{K}}, \bar{t}^{\mathcal{V}}\right)^2 = \left(c(t^{\mathcal{K}}, \bar{t}^{\mathcal{V}}.t^{\mathcal{K}}), t^{\mathcal{K}} + \bar{t}^{\mathcal{V}}.t^{\mathcal{K}}, (\bar{t}^{\mathcal{V}})^2\right) \\
&= \left(-c(t^{\mathcal{K}}, t^{\mathcal{K}}), 0, 1\right) = (0, 0, 1) \qquad \text{and} \\
t^{\mathcal{X}}.s^{\mathcal{X}} &= t^{\mathcal{X}} s^{\mathcal{X}} t^{\mathcal{X}} = (0, t^{\mathcal{K}}, \bar{t}^{\mathcal{V}})(0, s^{\mathcal{K}}, \bar{s}^{\mathcal{V}})(0, t^{\mathcal{K}}, \bar{t}^{\mathcal{V}}) \\
&= \left(c(t^{\mathcal{K}}, \bar{t}^{\mathcal{V}}.s^{\mathcal{K}}), t^{\mathcal{K}} + \bar{t}^{\mathcal{V}}.s^{\mathcal{K}}, \bar{t}^{\mathcal{V}}\bar{s}^{\mathcal{V}}\right)(0, t^{\mathcal{K}}, \bar{t}^{\mathcal{V}}) \\
&= \left(c(\bar{t}^{\mathcal{V}}.t^{\mathcal{K}}, s^{\mathcal{K}}) + c(t^{\mathcal{K}} + \bar{t}^{\mathcal{V}}.s^{\mathcal{K}}, (\bar{t}^{\mathcal{V}}\bar{s}^{\mathcal{V}}).t^{\mathcal{K}}), \underbrace{*, *}_{(t.s)^{\mathcal{A}}}\right) \\
&= \left(c(-t^{\mathcal{K}}, s^{\mathcal{K}}) + c(s^{\mathcal{V}}.t^{\mathcal{K}} - s^{\mathcal{K}}, t^{\mathcal{K}}), (t.s)^{\mathcal{K}}, \overline{t.s}^{\mathcal{V}}\right) \\
&= \left(c(s^{\mathcal{V}}.t^{\mathcal{K}}, t^{\mathcal{K}}), (t.s)^{\mathcal{K}}, \overline{t.s}^{\mathcal{V}}\right).
\end{aligned}
$$

So, axioms (G3) and (G4) are satisfied if and only if $c(s^{\mathcal{V}}.t^{\mathcal{K}}, t^{\mathcal{K}}) = 0$. ∎

Remark 5.3 Let $\mathcal{X}$ be an arbitrary T-reflection group and let $\zeta : \mathcal{X} \to \mathcal{A}$ be a reflection morphism. We have the central extension $\zeta : \mathcal{K}^{\mathcal{X}} \to \mathcal{K}$. There is a unique map $c_{\mathcal{X}} : \mathcal{K}^2 \to (\mathcal{K}^{\mathcal{X}})'$ satisfying $c_{\mathcal{X}}\left(\zeta(k), \zeta(l)\right) = [k, l]$ for all k and $l \in \mathcal{K}$. This map is bilinear since $\mathcal{K}^{\mathcal{X}}$ is two-step nilpotent. It is also $\mathcal{V}$-invariant and alternating. $\diamond$

Suppose $\mathcal{X} = \mathcal{A}^b$ for a reflection bihomomorphism $b : \mathcal{K}^2 \to \mathcal{Z}$.

Lemma 5.4 *There is an injective group homomorphism $(\mathcal{K}^{\mathcal{X}})' \to 2\mathcal{Z}$ such that the following diagram commutes:*

$$
\begin{array}{ccc}
\mathcal{K}^2 & \xrightarrow{\ c_{\mathcal{X}}\ } & (\mathcal{K}^{\mathcal{X}})' \\
{\scriptstyle b}\searrow & & \downarrow \\
& \mathcal{Z} \xrightarrow{\hspace{2em}} & 2\mathcal{Z}
\end{array}
$$

Proof. For (z_1, k_1) and $(z_2, k_2) \in \mathcal{K}^{\mathcal{X}}$, we can compute the commutator in $\mathcal{K}^{\mathcal{X}}$ as follows:

$$[(z_1, k_1), (z_2, k_2)] = (z_1, k_1)(z_2, k_2)(-z_1, -k_1)(-z_2, -k_2)$$
$$= (z_1 + z_2 + c(k_1, k_2), k_1 + k_2)(-z_1 - z_2 + c(-k_1, -k_2), -k_1 - k_2)$$
$$= (2c(k_1, k_2), 0).$$
∎

Corollary 5.5 *If $\mathcal{Z}$ contains no involutions, then there is an injective group homomorphism $(\mathcal{K}^{\mathcal{X}})' \to \mathcal{Z}$ such that the following diagram commutes:*

$$\mathcal{K}^2 \longrightarrow (\mathcal{K}^{\mathcal{X}})'$$
$$\searrow \quad \downarrow$$
$$\mathcal{Z}.$$

Now let $\mathcal{X}$ be an arbitrary T-reflection group. The following Proposition is key observation that makes a connection between the bihomomorphism $c_{\mathcal{X}}$ and reflection bihomomorphisms.

Proposition 5.6 *If $\mathcal{A}$ separates reflections, then the map $c_{\mathcal{X}}$ is a reflection bihomomorphism.*

Proof. The map $c_{\mathcal{X}}$ is bihomomorphic since $\mathcal{K}^{\mathcal{X}}$ is 2-step nilpotent. It is alternating and $\mathcal{V}$-invariant. Now let s and $t \in T$. Since $\mathcal{K}$ is abelian, we have

$$\dot{s}^{\mathcal{A}} = [t^{\mathcal{K}}, s^{\mathcal{V}}.t^{\mathcal{K}}]\dot{s}^{\mathcal{A}} = [t^{\mathcal{A}}\dot{t}^{\mathcal{A}}, \dot{s}^{\mathcal{A}}.(t^{\mathcal{A}}\dot{t}^{\mathcal{A}})]\dot{s}^{\mathcal{A}}$$
$$= t^{\mathcal{A}}\dot{t}^{\mathcal{A}}\dot{s}^{\mathcal{A}}t^{\mathcal{A}}\dot{t}^{\mathcal{A}}\dot{s}^{\mathcal{A}}\dot{t}^{\mathcal{A}}t^{\mathcal{A}}\dot{s}^{\mathcal{A}}\dot{t}^{\mathcal{A}}t^{\mathcal{A}}\dot{s}^{\mathcal{A}}\dot{s}^{\mathcal{A}} = (t^{\mathcal{A}}\dot{t}^{\mathcal{A}}\dot{s}^{\mathcal{A}}t^{\mathcal{A}}\dot{t}^{\mathcal{A}}).\dot{s}^{\mathcal{A}}.$$

Since $\mathcal{A}$ separates reflections, this entails $\dot{s} = (t^{\mathcal{A}}\dot{t}^{\mathcal{A}}\dot{s}^{\mathcal{A}}t^{\mathcal{A}}\dot{t}^{\mathcal{A}}).\dot{s}$. Using Remark 2.6 we obtain

$$c_{\mathcal{X}}(t^{\mathcal{K}}, s^{\mathcal{V}}.t^{\mathcal{K}}) = [t^{\mathcal{X}}\dot{t}^{\mathcal{X}}, \hat{s}.(t^{\mathcal{X}}\dot{t}^{\mathcal{X}})] = ((t^{\mathcal{X}}\dot{t}^{\mathcal{X}}\dot{s}^{\mathcal{X}}t^{\mathcal{X}}\dot{t}^{\mathcal{X}}).\dot{s}^{\mathcal{X}})\dot{s}^{\mathcal{X}}$$
$$= ((t^{\mathcal{X}}\dot{t}^{\mathcal{X}}\dot{s}^{\mathcal{X}}t^{\mathcal{X}}\dot{t}^{\mathcal{X}}).\dot{s})^{\mathcal{X}}\dot{s}^{\mathcal{X}} = ((t^{\mathcal{A}}\dot{t}^{\mathcal{A}}\dot{s}^{\mathcal{A}}t^{\mathcal{A}}\dot{t}^{\mathcal{A}}).\dot{s})^{\mathcal{X}}\dot{s}^{\mathcal{X}} = \dot{s}^{\mathcal{X}}\dot{s}^{\mathcal{X}} = 1.$$

We are done by Lemma 5.2. ∎

6 Finite root systems

In this section we investigate the action of the Weyl group $\mathcal{V}$ of an irreducible finite root system Δ on the root lattice $\mathcal{L}$ and the coroot lattice $\mathcal{L}^{\vee}$. We will derive results needed in the last section of this article. All of these results are stated in terms of the action of $\mathcal{V}$ on the abelian groups $\mathcal{L}$ and $\mathcal{L}^{\vee}$ and their subsets Δ and $\Delta^{\vee}$ (the coroot system) and rely on the pairing $\mathcal{L}^{\vee} \times \mathcal{L} \to \mathbb{Z}$ and the bijection $\Delta \to \Delta^{\vee}$.

Type	Dynkin Diagram	$\langle \alpha^\vee, \beta \rangle$	$\langle \beta^\vee, \alpha \rangle$	$r_\alpha.\beta$	$r_\beta.\alpha$
A_2	$\alpha \circ\!\!\!-\!\!\!-\!\!\!-\!\!\!\circ \beta$	-1	-1	$\beta + \alpha$	$\alpha + \beta$
B_2	$\alpha \circ\!\!\!=\!\!\!=\!\!\!\Leftarrow\!\!\!\circ \beta$	-2	-1	$\beta + 2\alpha$	$\alpha + \beta$
G_2	$\alpha \circ\!\!\!\equiv\!\!\!\Leftarrow\!\!\!\circ \beta$	-3	-1	$\beta + 3\alpha$	$\alpha + \beta$

Table 1: Irreducible finite root systems of rank 2

Throughout this section, let Δ be an irreducible finite root system with root basis B. If α and β are distinct roots in B, then we will say that α and β are *adjacent* and write $\alpha \sim \beta$ if they are connected by an edge in the Dynkin diagram. We will say that α and β are *simply adjacent* if they are connected by a simple edge. We will write $\alpha \not\sim \beta$ if α and β are not connected by an edge.

The root system Δ is the disjoint union of the divisible roots Δ_{ex} (for extra long) and the indivisible ones Δ_{red} (for reduced). The set Δ_{red}, in turn, can be partitioned into the short roots Δ_{sh} and the long roots Δ_{lg}. The coroots $\Delta^\vee$ form a root system in their own right and we partition them in the same way: $\Delta^\vee = \Delta_{\text{sh}}^\vee \cup \Delta_{\text{lg}}^\vee \cup \Delta_{\text{ex}}^\vee$. Denote the root lattice by $\mathcal{L}$ and the coroot lattice by $\mathcal{L}^\vee$. There is a bihomomorphic pairing $\mathcal{L}^\vee \times \mathcal{L} \to \mathbb{Z}$, $(\lambda, \mu) \mapsto \langle \lambda, \mu \rangle$ and a bijection $\bullet^\vee$, $\Delta \to \Delta^\vee$, $\alpha \mapsto \alpha^\vee$. For every $\alpha \in \Delta$ there is a reflection r_α that acts in the following ways on $\mathcal{L}$ and $\mathcal{L}^\vee$:

$$r_\alpha.\lambda = \lambda - \langle \alpha^\vee, \lambda \rangle \alpha \quad \text{and} \quad r_\alpha.\mu = \mu - \langle \mu, \alpha \rangle \alpha^\vee$$

for $\lambda \in \mathcal{L}$ and $\mu \in \mathcal{L}^\vee$. The Weyl group $\mathcal{V}$ is the subgroup of $\text{Aut}(\mathcal{L})$ generated by the reflections r_Δ. It acts faithfully and by group automorphisms on $\mathcal{L}$ and $\mathcal{L}^\vee$. We can view $\mathcal{L}$ and $\mathcal{L}^\vee$ as $\mathcal{V}$-modules. The bijection $\Delta \to \Delta^\vee$ is $\mathcal{V}$-equivariant and the pairing $\langle \cdot, \cdot \rangle$ is $\mathcal{V}$-invariant in the following sense: We have $\langle v.\lambda, v.\mu \rangle = \langle \lambda, \mu \rangle$ for $\lambda \in \mathcal{L}$, $\mu \in \mathcal{L}^\vee$ and $v \in \mathcal{V}$. We provide some important values of the pairing for root systems of rank 2 in Table 1 using the standard terminology of the classification of the irreducible finite root systems.

If we set $T := \{ r_\alpha \mid \alpha \in \Delta \}$, then T is a symmetric system with the multiplication defined by $t.s = tst^{-1}$ for s and $t \in T$.

Definition 6.1 We call T *the symmetric system associated to* Δ.   ⋄

If $S := \{ r_\alpha \mid \alpha \in B \}$, then $(\mathcal{V}, S)$ is a Coxeter system. As discussed in Example 2.13, the group $\mathcal{V}$ is the initial T-reflection group. Recall that for α and $\beta \in \Delta$ we write $r_\alpha \perp r_\beta$ if $r_\alpha \neq r_\beta$ and $[r_\alpha, r_\beta] = 1$.

Remark 6.2 If Δ is reduced and simply laced, i.e. $\Delta_{\text{lg}} = \Delta_{\text{ex}} = \varnothing$, then the map $\Delta \to \Delta^\vee$ extends to a $\mathcal{V}$-equivariant group isomorphism $\mathcal{L} \to \mathcal{L}^\vee$. So we can identify the $\mathcal{V}$-modules $\mathcal{L}$ and $\mathcal{L}^\vee$. If Δ is non-reduced, i.e. $\Delta_{\text{ex}} \neq \varnothing$, then there is an $\mathcal{L}$-equivariant isomorphism $\mathcal{L} \to \mathcal{L}^\vee$ that maps Δ in the following

way:

$$\alpha \mapsto \begin{cases} \alpha^\vee & \text{if } \alpha \in \Delta_{\text{lg}}, \\ \frac{1}{2}\alpha^\vee & \text{if } \alpha \in \Delta_{\text{sh}}, \\ 2\alpha^\vee & \text{if } \alpha \in \Delta_{\text{ex}}. \end{cases}$$

Again, we may identify the $\mathcal{V}$-modules $\mathcal{L}$ and $\mathcal{L}^\vee$. $\diamond$

Now suppose Δ is reduced and non-simply laced, i.e. $\Delta_{\text{ex}} = \varnothing$ and $\Delta_{\text{lg}} \neq \varnothing$. Then Δ_{sh} is mapped to $\Delta_{\text{lg}}^\vee$ and Δ_{lg} is mapped to $\Delta_{\text{sh}}^\vee$ by $\Delta \to \Delta^\vee$. Set

$$k_\Delta := \begin{cases} 2 & \text{if } \Delta \text{ is of type } B_\ell(\ell \geq 2), \ C_\ell(\ell \geq 3), \ F_4 \text{ or } BC_\ell(\ell \geq 2), \\ 3 & \text{if } \Delta \text{ is of type } G_2. \end{cases} \quad (4)$$

Lemma 6.3 *There is an injective $\mathcal{V}$-equivariant group homomorphism from $\mathcal{L} \to \mathcal{L}^\vee$ satisfying*

$$\varphi(\alpha) = \begin{cases} \alpha^\vee & \text{if } \alpha \in \Delta_{\text{lg}} \\ k_\Delta \alpha^\vee & \text{if } \alpha \in \Delta_{\text{sh}} \end{cases}$$

Via φ the $\mathcal{V}$-module $\mathcal{L}$ can be identified with the submodule of $\mathcal{L}^\vee$ generated by $\Delta_{\text{sh}}^\vee$. With this identification we have $k_\Delta \mathcal{L}^\vee \subseteq \mathcal{L} \subseteq \mathcal{L}^\vee$.

Proof. The groups $\mathcal{L}$ and $\mathcal{L}^\vee$ can be viewed as subgroups of a real vector space V and its dual V^* respectively. According to [Bou68] Ch. VI, n° 1.1 Proposition 3 and [Bou68] Ch. VI, n° 1.4 Proposition 11 we can pick a $\mathcal{V}$-invariant symmetric non-degenerate bilinear form $(\bullet|\bullet)$ on V that satisfies $(\beta, \beta) = 2$ for all $\beta \in \Delta_{\text{lg}}$. If we use this form to identify V with V^*, then we have

$$\alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)}$$

for every $\alpha \in \Delta$. If $\alpha \in \Delta_{\text{sh}}$ and $\beta \in \Delta_{\text{lg}}$ then

$$\frac{(\beta|\beta)}{(\alpha|\alpha)} = \frac{\langle \alpha^\vee, \beta \rangle}{\langle \beta^\vee, \alpha \rangle} = k_\Delta, \quad \text{so} \quad \alpha^\vee = \frac{2\alpha}{(\alpha|\alpha)} = k_\Delta \alpha \quad \text{and} \quad \beta^\vee = \frac{2\beta}{(\beta|\beta)} = \beta.$$

The last statement in the Lemma follows from the fact that the root lattice of a root system is generated by the short roots. ∎

Remark 6.4 If Δ is non-reduced, i.e. is of type $BC_\ell(\ell \geq 1)$ then Δ_{red} is a reduced irreducible root system. (See for instance [Bou68] Ch. VI, n° 4.14.) The Weyl group and the root lattice remain unchanged when passing from Δ to Δ_{red} and the following changes in types can occur:

$$BC_\ell \mapsto \begin{cases} A_1 & \text{if } \ell = 1 \\ B_\ell & \text{if } \ell \geq 2. \end{cases} \qquad \diamond$$

Set
$$\mathcal{L}_{\mathrm{eff}} = \langle v.l - l \mid v \in \mathcal{V}, l \in \mathcal{L}\rangle.$$

Proposition 6.5

$$\frac{\mathcal{L}}{\mathcal{L}_{\mathrm{eff}}} \cong \begin{cases} \mathbb{Z}_2 & \textit{if } \Delta \textit{ is of type } A_1, \ B_\ell(\ell \geq 2) \textit{ or } BC_\ell(\ell \geq 1), \\ \{0\} & \textit{otherwise.} \end{cases}$$

In the case of $\mathcal{L}/\mathcal{L}_{\mathrm{eff}} \cong \mathbb{Z}_2$ the quotient map $\mathcal{L} \to \mathbb{Z}_2$ satisfies

$$\alpha \mapsto \begin{cases} 1 & \textit{if } \alpha \textit{ is short or the type is } A_1 \\ 0 & \textit{otherwise} \end{cases}$$

for every $\alpha \in \Delta$.

Proof. In view of Remark 6.4, it suffices to consider reduced root systems Δ. The following is true if Δ is a root system of any type except A_1 or $B_\ell(\ell \geq 2)$: If $\alpha \in B$ then there is a root $\beta \in R$ such that $r_\alpha.\beta = \beta + \alpha$, i.e. $\langle \alpha^\vee, \beta \rangle = -1$. We will show this by using the classification of the finite irreducible root systems and the information from Table 1. In the cases $A_\ell(\ell \geq 2)$, $D_\ell(\ell \geq 4)$, E_6, E_7, E_8 and F_4 every vertex of the Dynkin diagram is connected to another vertex by a simple edge. So we can take β to be in B and simply adjacent to α. In the case of type $C_\ell(\ell \geq 3)$ and α long we may use the only adjacent root β. To understand the case G_2, let $B = \{\alpha, \beta\}$ with α short and β long. Then we have

$$\begin{aligned} r_\alpha.(\beta + \alpha) &= \beta + 2\alpha \quad \text{and} \\ r_\beta.\alpha &= \alpha + \beta. \end{aligned}$$

We may conclude $\mathcal{L}_{\mathrm{eff}} = \mathcal{L}$ and thus $\mathcal{L}/\mathcal{L}_{\mathrm{eff}} = \{0\}$.

Now we look at the construction of B_ℓ given in [Bou68] Ch. VI, n^0 4.5. By allowing $\ell = 1$ we also cover the case of A_1. So let $\epsilon_1, \dots \epsilon_\ell$ be a basis of $\mathbb{Z}^n$. The short roots in Δ are $\pm\epsilon_i$ $(i = 1, \dots, \ell)$ and the long roots are $\pm\epsilon_i \pm \epsilon_j$ $(1 \leq i < j \leq \ell)$. The root lattice $\mathcal{L}$ is $\mathbb{Z}^n$. According to [Bou68] Ch. VI, n^0 1.4 Proposition 11, the Weyl group $\mathcal{V}$ acts transitively on the short roots and on the long roots. So

$$\mathcal{L}_{\mathrm{eff}} = \left\{ \sum_{i=1}^{\ell} a_i\epsilon_i \ \Big| \ a_1, \dots, a_\ell \in \mathbb{Z}, \ \sum_{i=1}^{\ell} a_i \equiv 0 \mod 2 \right\}$$

This entails $\mathcal{L}/\mathcal{L}_{\mathrm{eff}} = \mathbb{Z}_2$. The second statement of the proposition is clear from the construction. ∎

Lemma 6.6 *There is a $\mathcal{V}$-invariant bihomomorphism $(\bullet|\bullet) : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$ with the following universal property: If $\beta : \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$ is a $\mathcal{V}$-invariant bihomomorphism, then there is a unique group homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}$ such that*

the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{L} \times \mathcal{L} & \xrightarrow{\;(\cdot,\cdot)\;} & \mathbb{Z} \\
& {\scriptstyle\beta}\searrow & \big\downarrow{\scriptstyle\varphi} \\
& & \mathbb{Z}.
\end{array}
$$

Moreover, the bihomomorphism $(\cdot|\cdot)$ *is symmetric and satisfies* $(\alpha|\beta) = 0$ *for* $\alpha, \beta \in \Delta$ *with* $r_\alpha \perp r_\beta$.

Proof. According to [Bou68] Ch. VI, nº 1.1 Proposition 3, there is a $\mathcal{V}$-invariant symmetric bihomomorphism $(\cdot|\cdot) : \ \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$. Since $(\mathcal{L}|\mathcal{L})$ is a non-trivial subgroup of $\mathbb{Z}$, it is possible to pass to a $\mathcal{V}$-invariant symmetric bihomomorphism $(\cdot,\cdot) : \ \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$ with $(\mathcal{L},\mathcal{L}) = \mathbb{Z}$. It satisfies $(\alpha|\beta) = 0$ for $\alpha, \beta \in \Delta$ with $r_\alpha \perp r_\beta$.

Denote the tensor product $\mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q}$ by $\mathcal{L}_{\mathbb{Q}}$ and suppose $\beta : \ \mathcal{L} \times \mathcal{L} \to \mathbb{Z}$ is a $\mathcal{V}$-invariant bihomomorphism. It can be extended to a $\mathcal{V}$-invariant $\mathbb{Q}$-bilinear form $\beta_{\mathbb{Q}} : \ \mathcal{L}_{\mathbb{Q}} \times \mathcal{L}_{\mathbb{Q}} \to \mathbb{Q}$. In the same way, we extend $(\cdot,\cdot)$ to $(\cdot,\cdot)_{\mathbb{Q}}$. According to [Bou68] Ch. V, nº 2.2 Proposition 1, the form $\beta_{\mathbb{Q}}$ is a $\mathbb{Q}$-multiple of $(\cdot,\cdot)_{\mathbb{Q}}$. Since $1 \in (\mathcal{L},\mathcal{L})$ and $\beta(\mathcal{L},\mathcal{L}) \subseteq \mathbb{Z}$, we have $\beta = k(\cdot,\cdot)$ for some $k \in \mathbb{Z}$. If we set $\varphi : \ \mathbb{Z} \to \mathbb{Z}, \ z \mapsto kz$ then the diagram in the lemma commutes. $\blacksquare$

Set

$$
\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L} := \frac{\mathcal{L} \otimes \mathcal{L}}{\big\langle (v.\lambda) \otimes (v.\mu) - \lambda \otimes \mu \mid v \in \mathcal{V}; \ \lambda, \mu \in \mathcal{L} \big\rangle}
$$

and denote by $\otimes_{\mathcal{V}} : \ \mathcal{L} \times \mathcal{L} \to \mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}$ the associated $\mathcal{V}$-invariant bihomomorphism.

Theorem 6.7 *We have*

$$
\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L} \cong
\begin{cases}
\mathbb{Z} \times \mathbb{Z}_2 & \text{if } \Delta \text{ is of type } B_\ell (\ell \geq 2) \text{ or } BC_\ell (\ell \geq 2) \\
\mathbb{Z} & \text{otherwise},
\end{cases}
$$

the bihomomorphism $\mathcal{L}^2 \to \mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}$ *is symmetric and* $\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}$ *is generated by the set*

$$
\{\alpha \otimes_{\mathcal{V}} \alpha \mid \alpha \in \Delta_{\mathrm{sh}}\} \cup \{\alpha \otimes_{\mathcal{V}} (r_\beta.\alpha) \mid \alpha \in \Delta_{\mathrm{sh}}, \beta \in \Delta\}.
$$

Before we begin the proof of this theorem we will establish two preliminary results.

Lemma 6.8 *The map* $\otimes_{\mathcal{V}}$ *is symmetric and* $\alpha \otimes_{\mathcal{V}} \beta = \beta \otimes_{\mathcal{V}} \alpha = 0$ *for every pair of roots* α *and* $\beta \in B$ *with* $\alpha \not\sim \beta$.

Proof. Let α and β be adjacent in B. Then Table 1 shows that $\langle \beta^\vee, \alpha \rangle = -1$ or $\langle \alpha^\vee, \beta \rangle = -1$. Without loss of generality we assume the latter. Then

$$
\beta \otimes_{\mathcal{V}} (\beta + \alpha) \ = \ \beta \otimes_{\mathcal{V}} (r_\alpha.\beta) \ = \ (r_\alpha.\beta) \otimes_{\mathcal{V}} \beta \ = \ (\beta + \alpha) \otimes_{\mathcal{V}} \beta.
$$

this entails $\alpha \otimes_V \beta = \beta \otimes_V \alpha$.

Now suppose α and $\beta \in B$ with $\alpha \not\sim \beta$. This entails $\langle \beta^\vee, \alpha \rangle = 0$. Since the Dynkin graph is connected and all except possibly one edge are simple there must be a root $\gamma \in B$ such that α and γ are simply adjacent, or β and γ are simply adjacent. We may assume the latter. So we have $\langle \beta^\vee, \gamma \rangle = -1$. Then

$$
\begin{aligned}
\alpha \otimes_V \gamma &= (r_\beta . \alpha) \otimes_V (r_\beta . \gamma) = \alpha \otimes_V (\gamma + \beta) \\
\gamma \otimes_V \alpha &= (r_\beta . \gamma) \otimes_V (r_\beta . \alpha) = (\gamma + \beta) \otimes_V \alpha,
\end{aligned}
$$

so, $\alpha \otimes_V \beta = \beta \otimes_V \alpha = 0$.   ■

Lemma 6.9 *Let α, β and γ be roots.*

(i) *If $\alpha \not\sim \gamma$ then $\langle \beta^\vee, \gamma \rangle \alpha \otimes_V \beta = \langle \beta^\vee, \alpha \rangle \beta \otimes_V \gamma$.*

(ii) *$2\alpha \otimes_V \beta = \langle \alpha^\vee, \beta \rangle \alpha \otimes_V \alpha$.*

Proof. Ad (i):
$$
\begin{aligned}
\langle \beta^\vee, \gamma \rangle \alpha \otimes_V \beta &= -\alpha \otimes_V (\gamma - \langle \beta^\vee, \gamma \rangle \beta) = -\alpha \otimes_V (r_\beta . \gamma) \\
&= -(r_\beta . \alpha) \otimes_V \gamma = -(\alpha - \langle \beta^\vee, \alpha \rangle \beta) \otimes_V \gamma \\
&= \langle \beta^\vee, \alpha \rangle \beta \otimes_V \gamma.
\end{aligned}
$$
Ad (ii): $\alpha \otimes_V \beta = (r_\alpha . \alpha) \otimes_V (r_\alpha . \beta) = -\alpha \otimes_V (\beta - \langle \alpha^\vee, \beta \rangle \alpha)$.   ■

Now we prove the theorem.

Proof. Due to Remark 6.4 we my assume that the root system Δ is reduced. Note that with the map $(\bullet|\bullet)$ from Lemma 6.6 and the universal property of $\otimes_V$ we have the following commuting diagram:

So in order to prove $\mathcal{L} \otimes_V \mathcal{L} \cong \mathbb{Z}$, it suffices to show that $\mathcal{L} \otimes_V \mathcal{L}$ is cyclic.

Case 1:   The root system Δ is of type A_1. Then $\mathcal{L} \cong \mathbb{Z}$, generated by the single root $\alpha \in B$. So $\mathcal{L} \otimes_V \mathcal{L}$ is generated by $\alpha \otimes_V \alpha$.

Case 2:   The root system Δ is simply laced and of rank $\ell \geq 2$, i.e. of one of the types $A_\ell(\ell \geq 2)$, $D_\ell(\ell \geq 4)$, E_6, E_7 and E_8. Let α and α' be adjacent roots in B. Lemma 6.9 (ii) implies $\alpha \otimes_V \alpha = \alpha' \otimes_V \alpha' = -2\alpha \otimes_V \alpha'$. Moreover, if $\alpha'' \in B$ is adjacent to α' then $\alpha \not\sim \alpha''$ since the Dynkin diagram contains no loops. Lemma 6.9 (i) implies $\alpha \otimes_V \alpha' = \alpha' \otimes_V \alpha''$. Since the Dynkin diagram is connected and in view of Lemma 6.8 we obtain the following for γ and $\delta \in B$:

$$
\gamma \otimes_V \delta = \begin{cases} \alpha \otimes_V \alpha' & \text{if } \gamma \text{ and } \delta \text{ are adjacent} \\ -2\alpha \otimes_V \alpha' & \text{if } \gamma = \delta \end{cases}
$$

Again in view of Lemma 6.8, this proves that $\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}$ is generated by

$$
\begin{aligned}
\alpha' \otimes_{\mathcal{V}} (r_\alpha.\alpha') \;&=\; \alpha' \otimes_{\mathcal{V}} (\alpha' + \alpha) \;=\; \alpha' \otimes_{\mathcal{V}} \alpha' + \alpha' \otimes_{\mathcal{V}} \alpha \\
&=\; -2\alpha \otimes_{\mathcal{V}} \alpha' + \alpha' \otimes_{\mathcal{V}} \alpha \;=\; -\,\alpha \otimes_{\mathcal{V}} \alpha'.
\end{aligned}
$$

In the following cases there will be two different root lengths. Let α and β be the unique pair of adjacent roots in B such that α is short and β is long.

Case 3: There is at least one short root $\alpha' \in B$ other than α, i.e. the root system Δ is of type $C_\ell(\ell \geq 3)$ or F_4. Note that we have $\langle \alpha^\vee, \beta \rangle = -2$ and $\langle \beta^\vee, \alpha \rangle = -1$. If γ and $\delta \in B$ then

$$
\gamma \otimes_{\mathcal{V}} \delta =
\begin{cases}
2\alpha \otimes_{\mathcal{V}} \alpha' & \text{if } \gamma = \alpha \text{ and } \delta = \beta & \text{(by Lemma 6.9 (ii))}, \\
\alpha \otimes_{\mathcal{V}} \alpha' & \text{if } \gamma \text{ and } \delta \in \Delta_{\text{sh}} \text{ and } \gamma \sim \delta & \text{(by Lemma 6.9 (ii))}, \\
2\alpha \otimes_{\mathcal{V}} \alpha' & \text{if } \gamma \text{ and } \delta \in \Delta_{\text{lg}} \text{ and } \gamma \sim \delta & \text{(by Lemma 6.9 (ii))}, \\
-2\alpha \otimes_{\mathcal{V}} \alpha' & \text{if } \gamma = \delta \in \Delta_{\text{sh}} & \text{(by Lemma 6.9 (i))}, \\
-4\alpha \otimes_{\mathcal{V}} \alpha' & \text{if } \gamma = \delta \in \Delta_{\text{lg}} & \text{(by Lemma 6.9 (i))}.
\end{cases}
$$

As in the previous case $\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}$ is generated by $\alpha' \otimes_{\mathcal{V}} (r_\alpha.\alpha')$.

Case 4: The root system Δ is of type G_2. Then we obtain

$$
\beta \otimes_{\mathcal{V}} \beta = -2\alpha \otimes_{\mathcal{V}} \beta \quad \text{and} \quad 3\alpha \otimes_{\mathcal{V}} \alpha = -2\alpha \otimes_{\mathcal{V}} \beta.
$$

The group $\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}$ is generated by the set $\{\alpha \otimes_{\mathcal{V}} \beta, \alpha \otimes_{\mathcal{V}} \alpha\}$. Thus the group $\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}$ is a quotient of the group $\mathbb{Z}^2/\big(\mathbb{Z}(2,3)\big) \cong \mathbb{Z}$. This is a cyclic group. Moreover, since

$$
\alpha \otimes_{\mathcal{V}} \beta \;=\; \alpha \otimes_{\mathcal{V}} (r_\beta.\alpha) - \alpha \otimes_{\mathcal{V}} \alpha
$$

the group is generated by the set $\{\alpha \otimes_{\mathcal{V}} (r_\beta.\alpha), \alpha \otimes_{\mathcal{V}} \alpha\}$.

Case 5: The root system Δ is of type $B_\ell(\ell \geq 2)$. If γ and $\delta \in B$ then

$$
\gamma \otimes_{\mathcal{V}} \delta =
\begin{cases}
\alpha \otimes_{\mathcal{V}} \beta & \text{if } \gamma \text{ and } \delta \text{ are adjacent and both of them are long} \\
-2\alpha \otimes_{\mathcal{V}} \beta & \text{if } \gamma = \delta \text{ and } \gamma \text{ is long.}
\end{cases}
$$

The group $\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}$ is generated by $\{\alpha \otimes_{\mathcal{V}} \beta, \alpha \otimes_{\mathcal{V}} \alpha\}$ and satisfies the relation $2\alpha \otimes_{\mathcal{V}} \beta = -2\alpha \otimes_{\mathcal{V}} \alpha$. As in the previous case, the set $\{\alpha \otimes_{\mathcal{V}} (r_\beta.\alpha), \alpha \otimes_{\mathcal{V}} \alpha\}$ is a generating set. But this time, the group $\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}$ is a quotient of the group $\mathbb{Z}^2/\big(\mathbb{Z}(2,2)\big) \cong \mathbb{Z} \times \mathbb{Z}_2$. To show that $\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}$ is actually isomorphic to this group, we will construct a surjective $\mathcal{V}$-invariant bihomomorphism $\mathcal{L} \times \mathcal{L} \to \mathbb{Z} \times \mathbb{Z}_2$. Note that the considerations above imply $-(\alpha|\beta) = (\alpha|\alpha) = 1$.

Due to Proposition 6.5 we have $\mathcal{L}/\mathcal{L}_{\text{eff}} \cong \mathbb{Z}_2$. Denote the quotient homomorphism by $\mathcal{L} \to \mathbb{Z}_2$, $\lambda \mapsto \overline{\lambda}$. Set

$$
b: \; \mathcal{L}^2 \to \mathbb{Z} \times \mathbb{Z}_2, \;\; (\lambda, \mu) \mapsto \big((\lambda|\mu), \overline{\lambda}\overline{\mu}\big).
$$

This is a $\mathcal{V}$-invariant bihomomorphism and it is surjective due to

$$
-b(\alpha, \beta) = (1,0) \quad \text{and} \quad b(\alpha, \alpha + \beta) = (0,1). \qquad \blacksquare
$$

Set

$$\mathcal{L} \boxtimes \mathcal{L} := \frac{\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}}{\langle \alpha \otimes_{\mathcal{V}} \beta \mid \alpha, \beta \in \Delta_{\mathrm{sh}} \text{ with } r_\alpha \perp r_\beta \rangle}$$

and denote by $\boxtimes : \mathcal{L} \times \mathcal{L} \to \mathcal{L} \boxtimes \mathcal{L}$ the associated $\mathcal{V}$-invariant bihomomorphism.

Corollary 6.10 *We have $\mathcal{L} \boxtimes \mathcal{L} \cong \mathbb{Z}$ and $\mathcal{L} \boxtimes \mathcal{L}$ is generated by the set*

$$\{\alpha \boxtimes \alpha \mid \alpha \in \Delta_{\mathrm{sh}}\} \cup \{\alpha \boxtimes (r_\beta.\alpha) \mid \alpha \in \Delta_{\mathrm{sh}}, \beta \in \Delta\}.$$

Moreover $\alpha \boxtimes \beta = 0$ for $\alpha, \beta \in \Delta$ with $r_\alpha \perp r_\beta$.

Proof. According to Lemma 6.6 we have $(\alpha|\beta) = 0$ for $\alpha, \beta \in \Delta$ with $r_\alpha \perp r_\beta$. So there is a homomorphism $\mathcal{L} \boxtimes \mathcal{L} \to \mathbb{Z}$ such that the following diagram is commutative:

We have to revisit Case 5 in the proof of the previous theorem. Consider the root $\omega := r_\beta.\alpha = \alpha + \beta$. Note that

$$\langle \alpha^\vee, \omega \rangle \;=\; \langle \alpha^\vee, \alpha \rangle + \langle \alpha^\vee, \beta \rangle \;=\; 0$$

This entails $\langle \omega^\vee, \alpha \rangle = 0$, so $[r_\alpha, r_\omega] = 1$. This entails the relation

$$0 \;=\; \alpha \boxtimes \omega \;=\; \alpha \boxtimes \alpha + \alpha \boxtimes \beta.$$

This means that $\mathcal{L} \boxtimes \mathcal{L}$ is cyclic. ∎

Set

$$\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}^\vee := \frac{\mathcal{L} \otimes \mathcal{L}^\vee}{\langle (v.\lambda) \otimes (v.\mu) - \lambda \otimes \mu \mid v \in \mathcal{V}, \; \lambda \in \mathcal{L}, \; \mu \in \mathcal{L}^\vee \rangle}$$

and denote by $\otimes_{\mathcal{V}} : \mathcal{L} \times \mathcal{L}^\vee \to \mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}^\vee$ the associated $\mathcal{V}$-invariant bihomomorphism.

Theorem 6.11 *We have*

$$\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}^\vee \cong \begin{cases} \mathbb{Z} \times \mathbb{Z}_2 & \text{if } \Delta \text{ is of type } BC_\ell(\ell \geq 2) \\ \mathbb{Z} & \text{otherwise.} \end{cases}$$

We will start with versions of Lemma 6.9 and Lemma 6.8 appropriate for elements of $\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}^\vee$.

Lemma 6.12 *Suppose α and $\beta \in B$. Then the following are satisfied:*

(i) *If α and β have the same length then $\alpha \otimes_{\mathcal{V}} \beta^{\vee} = \beta \otimes_{\mathcal{V}} \alpha^{\vee}$.*

(ii) *If $\alpha \not\sim \beta$ then $\alpha \otimes_{\mathcal{V}} \beta^{\vee} = \beta \otimes_{\mathcal{V}} \alpha^{\vee} = 0$.*

Proof.   Let α and β be adjacent in B and of the same length. Then Table 1 shows that $\langle \beta^{\vee}, \alpha \rangle = \langle \alpha^{\vee}, \beta \rangle = -1$. So,

$$\beta \otimes_{\mathcal{V}} (\beta^{\vee} + \alpha^{\vee}) \;\; = \;\; \beta \otimes_{\mathcal{V}} (r_{\alpha}.\beta^{\vee}) \;\; = \;\; (r_{\alpha}.\beta) \otimes_{\mathcal{V}} \beta^{\vee} \;\; = \;\; (\beta + \alpha) \otimes_{\mathcal{V}} \beta^{\vee}.$$

This entails $\alpha \otimes_{\mathcal{V}} \beta^{\vee} = \beta \otimes_{\mathcal{V}} \alpha^{\vee}$.

Now suppose α and $\beta \in B$ with $\alpha \not\sim \beta$. This entails $\langle \alpha^{\vee}, \beta \rangle = \langle \beta^{\vee}, \alpha \rangle = 0$. Since the Dynkin graph is connected and all except possibly one edge are simple, there must be a root $\gamma \in B$ such that α and γ are simply adjacent, or β and γ are simply adjacent. We may assume the latter. So we have $\langle \gamma^{\vee}, \beta \rangle = -1$. Then

$$\alpha \otimes_{\mathcal{V}} \gamma^{\vee} \;\; = \;\; (r_{\beta}.\alpha) \otimes_{\mathcal{V}} (r_{\beta}.\gamma^{\vee}) \;\; = \;\; \alpha \otimes_{\mathcal{V}} (\gamma^{\vee} + \beta^{\vee}) \quad \text{and}$$
$$\gamma \otimes_{\mathcal{V}} \alpha^{\vee} \;\; = \;\; (r_{\beta}.\gamma) \otimes_{\mathcal{V}} (r_{\beta}.\alpha^{\vee}) \;\; = \;\; (\gamma + \beta) \otimes_{\mathcal{V}} \alpha^{\vee},$$

so, $\alpha \otimes_{\mathcal{V}} \beta^{\vee} = \beta \otimes_{\mathcal{V}} \alpha^{\vee} = 0$.   ∎

Lemma 6.13 *Let α, β and γ be roots.*

(i) *If $\alpha \not\sim \gamma$ then $\langle \gamma^{\vee}, \beta \rangle \alpha \otimes_{\mathcal{V}} \beta^{\vee} = \langle \beta^{\vee}, \alpha \rangle \beta \otimes_{\mathcal{V}} \gamma^{\vee}$.*

(ii) *$2\alpha \otimes_{\mathcal{V}} \beta^{\vee} = \langle \beta^{\vee}, \alpha \rangle \alpha \otimes_{\mathcal{V}} \alpha^{\vee} = \langle \beta^{\vee}, \alpha \rangle \beta \otimes_{\mathcal{V}} \beta^{\vee}$.*

Proof.   Ad (i):
$$\langle \gamma^{\vee}, \beta \rangle \alpha \otimes_{\mathcal{V}} \beta^{\vee} \;\; = \;\; -\alpha \otimes_{\mathcal{V}} (\gamma^{\vee} - \langle \gamma^{\vee}, \beta \rangle \beta^{\vee}) \;\; = \;\; -\alpha \otimes_{\mathcal{V}} (r_{\beta}.\gamma^{\vee})$$
$$= \;\; -(r_{\beta}.\alpha) \otimes_{\mathcal{V}} \gamma^{\vee} \;\; = \;\; -(\alpha - \langle \beta^{\vee}, \alpha \rangle \beta) \otimes_{\mathcal{V}} \gamma^{\vee}$$
$$= \;\; \langle \beta^{\vee}, \alpha \rangle \beta \otimes_{\mathcal{V}} \gamma^{\vee}.$$
Ad (ii): $\alpha \otimes_{\mathcal{V}} \beta^{\vee} \;\; = \;\; (r_{\alpha}.\alpha) \otimes_{\mathcal{V}} (r_{\alpha}.\beta^{\vee}) \;\; = \;\; -\alpha \otimes_{\mathcal{V}} (\beta^{\vee} - \langle \beta^{\vee}, \alpha \rangle \alpha^{\vee}),$
$\;\; \alpha \otimes_{\mathcal{V}} \beta^{\vee} \;\; = \;\; (r_{\beta}.\alpha) \otimes_{\mathcal{V}} (r_{\beta}.\beta^{\vee}) \;\; = \;\; (\alpha - \langle \beta^{\vee}, \alpha \rangle \beta) \otimes_{\mathcal{V}} (-\beta^{\vee}).$ ∎

Now we present the proof of the theorem.

Proof.   If the root system is reduced and simply laced, or if it is non-reduced, then we can identify the $\mathcal{V}$-modules $\mathcal{L}$ and $\mathcal{L}^{\vee}$ by Remark 6.2. In those cases, we are done by the previous theorem. So we focus on the non-simply laced reduced case:

We use the identification $\mathcal{L} \subseteq \mathcal{L}^{\vee}$ from Lemma 6.3. Due to the universal property of $\mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}^{\vee}$ we have the following commutative diagram:

$$
\begin{array}{ccc}
\mathcal{L} \times \mathcal{L}^{\vee} & \longrightarrow & \mathcal{L} \otimes_{\mathcal{V}} \mathcal{L}^{\vee} \\
& \searrow & \downarrow \\
& & \mathcal{L}^{\vee} \otimes_{\mathcal{V}} \mathcal{L}^{\vee}.
\end{array}
$$

Since $k_\Delta \mathcal{L}^\vee \subseteq \mathcal{L}$ the bihomomorphism $\mathcal{L} \times \mathcal{L}^\vee \to \mathcal{L}^\vee \otimes_\mathcal{V} \mathcal{L}^\vee$ in this diagram has $k_\Delta \mathcal{L}^\vee \otimes_\mathcal{V} \mathcal{L}^\vee$ and thus a copy of $\mathbb{Z}$ in its image. So for our proof it suffices to show that $\mathcal{L} \otimes_\mathcal{V} \mathcal{L}^\vee$ is cyclic.

Let α and β be the unique pair of adjacent roots in B such that α is short and β is long and recall that $k = -\langle \alpha^\vee, \beta \rangle$. Then

$$
\begin{aligned}
\beta \otimes_\mathcal{V} (\beta^\vee + \alpha^\vee) &= \beta \otimes_\mathcal{V} (r_\alpha . \beta^\vee) = (r_\alpha . \beta) \otimes_\mathcal{V} \beta^\vee = (\beta + k_\Delta \alpha) \otimes_\mathcal{V} \beta^\vee, \text{ so} \\
\beta \otimes_\mathcal{V} \alpha^\vee &= k_\Delta \alpha \otimes_\mathcal{V} \beta^\vee.
\end{aligned}
$$

Moreover, by Lemma 6.13 (ii), we have

$$
\alpha \otimes_\mathcal{V} \alpha^\vee = \beta \otimes_\mathcal{V} \beta^\vee = -2\alpha \otimes_\mathcal{V} \beta^\vee.
$$

If there is a root $\alpha' \in B$ with $\alpha' \sim \alpha$, i.e. if the root system Δ is of type $C_\ell(\ell \geq 3)$ or F_4, then α' is short, we have $\alpha' \not\sim \beta$ and

$$
\alpha' \otimes_\mathcal{V} \alpha^\vee = \alpha \otimes_\mathcal{V} \beta^\vee
$$

by Lemma 6.13 (i).

If there is a root $\beta' \in B$ with $\beta' \sim \beta$, i.e. if the root system Δ is of type $B_\ell(\ell \geq 3)$ or F_4, then β' is long, we have $\alpha \not\sim \beta'$ and

$$
\beta \otimes_\mathcal{V} (\beta')^\vee = \alpha \otimes_\mathcal{V} \beta^\vee
$$

by Lemma 6.13 (i).

So if γ and $\delta \in B$ then

$$
\gamma \otimes_\mathcal{V} \delta = \begin{cases}
k_\Delta \alpha \otimes_\mathcal{V} \beta^\vee & \text{if } \gamma = \beta \text{ and } \delta = \alpha, \\
\alpha \otimes_\mathcal{V} \beta^\vee & \text{if } \gamma, \delta \in \Delta_{\mathrm{sh}} \text{ and } \gamma \sim \delta \text{ (due to Lemma 6.9 (i))}, \\
\alpha \otimes_\mathcal{V} \beta^\vee & \text{if } \gamma, \delta \in \Delta_{\mathrm{lg}} \text{ and } \gamma \sim \delta \text{ (due to Lemma 6.9 (i))}, \\
-2\alpha \otimes_\mathcal{V} \beta^\vee & \text{if } \gamma = \delta.
\end{cases}
$$

This shows that $\mathcal{L} \otimes_\mathcal{V} \mathcal{L}^\vee$ is generated by $\alpha \otimes_\mathcal{V} \beta^\vee$ and is thus cyclic. ∎

Set

$$
\mathcal{L} \boxtimes \mathcal{L}^\vee := \frac{\mathcal{L} \otimes_\mathcal{V} \mathcal{L}^\vee}{\langle \alpha \otimes_\mathcal{V} \beta^\vee \mid \alpha, \beta \in \Delta \text{ with } r_\alpha \perp r_\beta \rangle}
$$

and denote by $\boxtimes : \mathcal{L} \times \mathcal{L}^\vee \to \mathcal{L} \boxtimes \mathcal{L}^\vee$ the associated bihomomorphism.

Remark 6.14 Suppose Δ is non-simply laced and reduced. With the identification $\mathcal{L} \subseteq \mathcal{L}^\vee$ from Lemma 6.3 we have the following commutative digram:

$$
\begin{array}{ccc}
\mathcal{L} \times \mathcal{L} & \longrightarrow & \mathcal{L} \boxtimes \mathcal{L} \\
\downarrow & & \downarrow{\scriptstyle \varphi} \\
\mathcal{L} \times \mathcal{L}^\vee & \longrightarrow & \mathcal{L} \boxtimes \mathcal{L}^\vee \\
\downarrow & & \downarrow{\scriptstyle \psi} \\
\mathcal{L}^\vee \times \mathcal{L}^\vee & \longrightarrow & \mathcal{L}^\vee \boxtimes \mathcal{L}^\vee.
\end{array}
\tag{5}
$$

The homomorphism φ exists due to the universal property of $\mathcal{L} \boxtimes \mathcal{L}$ and the homomorphism ψ exists due to the universal property of $\mathcal{L} \boxtimes \mathcal{L}^\vee$. ◇

Corollary 6.15 $\mathcal{L} \boxtimes \mathcal{L}^\vee \cong \mathbb{Z}$.

Proof. In the simply laced reduced case, and in the non-reduced case this follows from Corollary 6.10 with the identification from Remark 6.2. In the non-simply laced reduced case, we know that $\mathcal{L} \boxtimes \mathcal{L}^\vee$ is isomorphic to a quotient of $\mathbb{Z}$ by Theorem 6.11. So it suffices to see that it is not trivial. Consider the homomorphism ψ in (5). Since $k_\Delta \mathcal{L}^\vee \subseteq \mathcal{L}$, its image contains $k_\Delta \mathcal{L}^\vee \boxtimes \mathcal{L}^\vee$, which is not trivial by Corollary 6.10. ∎

Corollary 6.16 *The maps φ and ψ in (5) are injective.*

Proof. Since each of the groups $\mathcal{L} \boxtimes \mathcal{L}$, $\mathcal{L} \boxtimes \mathcal{L}^\vee$ and $\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee$ is isomorphic to $\mathbb{Z}$, this follows from the fact that neither φ nor ψ is trivial. ∎

7 Root systems extended by an abelian group

In this section we use the results derived in the previous one about a finite root system Δ to understand a root system R of the kind Δ extended by an abelian group G. (Compare [Yos04].) We associate a symmetric system T to R and a particular T-reflection group $\mathcal{W}$, the Weyl group of R. This notion of a Weyl group generalizes the notion of extended affine Weyl groups (EAWeGs). We prove the main result of this article (Theorem 7.20), which characterizes the relationship between the initial T-reflection group $\mathcal{U}$ and the Weyl group $\mathcal{W}$ in terms of their abelianizations.

We continue to work with the notation of the previous section, so let Δ be an irreducible finite root system with coroot system $\Delta^\vee$. Denote its Weyl group by $\mathcal{V}$, its root lattice by $\mathcal{L}$ and its coroot lattice by $\mathcal{L}^\vee$. Let G be an abelian group.

For $(g, \alpha) \in G \times \Delta$ set

$$r_{(g,\alpha)}: \ G \times \mathcal{L} \to G \times \mathcal{L}, \ (h, \beta) \mapsto (h - \langle \alpha^\vee, \beta \rangle g, r_\alpha.\beta).$$

Then $r_{(g,\alpha)}$ is a group homomorphism. Moreover, we have

$$
\begin{aligned}
(r_{(g,\alpha)} \circ r_{(g,\alpha)})(h, \beta) &= r_{(g,\alpha)}(h - \langle \alpha^\vee, \beta \rangle g, r_\alpha.\beta) \\
&= (h - \langle \alpha^\vee, \beta \rangle g - \langle \alpha^\vee, r_\alpha.\beta \rangle g, r_\alpha^2.\beta) \\
&= (h - \langle \alpha^\vee, \beta \rangle g - \langle \alpha^\vee, \beta \rangle (1 - \langle \alpha^\vee, \alpha \rangle) g, \beta) \\
&= (h, \beta) \qquad\qquad (6)
\end{aligned}
$$

for $(h, \beta) \in G \times \mathcal{L}$. In particular $r_{(\alpha,g)}$ is an element of the group

$$\mathrm{Aut}_G(G \times \mathcal{L}) = \{\varphi \in \mathrm{Aut}(G \times \mathcal{L}) \mid \varphi(g, 0) = (g, 0) \text{ for all } g \in G\}.$$

Definition 7.1 We shall call a subset $R \subseteq G \times \Delta$ a *root system of the kind* Δ *extended by* G if the following are satisfied:

(R0) The projection $R \to \Delta$ is surjective.

(R1) The image of the projection $R \to G$ generates G.

(R2) $\{0\} \times \Delta^{\mathrm{red}} \subseteq R$.

(R3) $r_\alpha.R \subseteq R$ for every $\alpha \in R$.

The *type* of the root system R is the type of Δ. $\diamond$

Remark 7.2 If we set

$$S_\alpha := \{g \in G \mid (g, \alpha) \in R\}$$

then each of the conditions (R0) to (R3) as equivalent to each of the following conditions, respectively:

(R0') For every $\alpha \in \Delta$ the set S_α is non-empty.

(R1') $\bigcup_{\alpha \in \Delta} S_\alpha$ generates G.

(R2') $0 \in S_\alpha$ for every $\alpha \in \Delta^{\mathrm{red}}$.

(R3') $S_\beta - \langle \alpha^\vee, \beta \rangle S_\alpha \subseteq S_{r_\alpha.\beta}$ for all α and $\beta \in \Delta$.

These are the axioms for a *(not necessarily reduced) root system of type* Δ *extended by* G defined in [Yos04]. So our definition coincides with the one given there. The anisotropic roots $R^\times$ of an extended affine root system (EARS) (see, for instance, [AAB$^+$97]) form a root system of the kind Δ extended by a finitely generated free abelian group G. So the results that follow are applicable to EARSs. $\diamond$

   We present some results about the sets S_α taken from [Yos04] Section 3. If two roots α and $\beta \in \Delta$ have the same length, then $S_\alpha = S_\beta$. This allows us to define S_{sh}, S_{lg} and S_{ex} as follows. Let $x \in \{\mathrm{sh}, \mathrm{lg}, \mathrm{ex}\}$. If Δ_x is not empty then $S_x := S_\alpha$ where $\alpha \in \Delta_x$. Provided the following exist we have

$$k_\Delta S_{\mathrm{lg}} \subseteq S_{\mathrm{ex}} \subseteq S_{\mathrm{lg}}, \quad k_\Delta S_{\mathrm{sh}} \subseteq S_{\mathrm{lg}} \subseteq S_{\mathrm{sh}}, \quad \text{and} \quad k_\Delta^2 S_{\mathrm{sh}} \subseteq S_{\mathrm{ex}} \subseteq S_{\mathrm{sh}}, \qquad (7)$$

where k_Δ is the number defined in (4) in the previous section. This means that condition (R1') (and thus condition (R1)) is equivalent to $\langle S_{\mathrm{sh}} \rangle = G$.

   From now on, let G be a torsion-free group and let R be a root system of the kind Δ extended by G.

Remark 7.3 The group $\text{Aut}_G(G \times \mathcal{L})$ is isomorphic to the semidirect product $\text{Hom}(L, G) \rtimes \text{Aut}(\mathcal{L})$, where we use the left action of $\text{Aut}(\mathcal{L})$ on $\text{Hom}(\mathcal{L}, G)$ given by $\varphi.\psi = \psi \circ \varphi^{-1}$ for $\varphi \in \text{Aut}(\mathcal{L})$ and $\psi \in \text{Hom}(\mathcal{L}, G)$. The isomorphism can be expressed by the action of $\text{Hom}(L, G) \rtimes \text{Aut}(\mathcal{L})$ on $G \times \mathcal{L}$ given by

$$(\varphi, \psi).(g, \lambda) = \big(g + \varphi \circ \psi(\lambda), \psi(\lambda)\big).$$

We will identify the two isomorphic groups.

There is an injective group homomorphism

$$\mathcal{L}^\vee \to \text{Hom}(\mathcal{L}, \mathbb{Z}), \ \lambda \mapsto (\mu \mapsto \langle \lambda, \mu \rangle)$$

This gives rise to a group homomorphism

$$G \otimes \mathcal{L}^\vee \to G \otimes \text{Hom}(\mathcal{L}, \mathbb{Z}) \cong \text{Hom}(\mathcal{L}, G),$$

which is injective since G is torsion-free. We will identify elements of $G \otimes \mathcal{L}$ with their corresponding images in $\text{Hom}(\mathcal{L}, G)$. With the two identifications discussed we have

$$r_{(g,\alpha)} = (g \otimes \alpha^\vee, r_\alpha)$$

for every $g \in G$ and $\alpha \in \Delta$. $\diamond$

Set
$$T := \{r_\alpha \mid \alpha \in R\} \subseteq \text{Aut}_G(G \times \mathcal{L}).$$

With the identification in Remark 7.3 we have

$$
\begin{aligned}
r(g, \alpha).r(h, \beta) &= r_{(g,\alpha)} r_{(h,\beta)} r_{(g,\alpha)} = (g \otimes \alpha^\vee, r_\alpha)(h \otimes \beta^\vee, r_\beta)(g \otimes \alpha^\vee, r_\alpha) \\
&= (g \otimes \alpha^\vee + h \otimes (r_\alpha.\beta^\vee), r_\alpha r_\beta)(g \otimes \alpha^\vee, r_\alpha) \\
&= \big(g \otimes \alpha^\vee + h \otimes (r_\alpha.\beta^\vee) + g \otimes (r_\alpha r_\beta.\alpha^\vee), r_\alpha r_\beta r_\alpha\big) \\
&= \big(h \otimes (r_\alpha.\beta^\vee) + g \otimes (\alpha^\vee - r_\alpha r_\beta r_\alpha.\alpha^\vee), r_{r_\alpha.\beta}\big) \\
&= \big(h \otimes (r_\alpha.\beta^\vee) + g \otimes (\langle \alpha^\vee, r_\alpha.\beta\rangle r_\alpha.\beta^\vee), r_{r_\alpha.\beta}\big) \\
&= \big((h - \langle \alpha^\vee, \beta\rangle g) \otimes (r_\alpha.\beta^\vee), r_{r_\alpha.\beta}\big) = r_{(h - \langle \alpha^\vee, \beta\rangle g, r_\alpha.\beta)} \\
&= r_{r_{(g,\alpha)}(h,\beta)}.
\end{aligned}
$$

So, for every s and $t \in T$ the element $s.t := sts^{-1}$ is in T and defines a multiplication $T \times T \to T$, $(s, t) \mapsto s.t$ on T. As observed in Example 2.2 this multiplication turns T into a symmetric system.

Definition 7.4 We call T *the symmetric system associated to R.* $\diamond$

Denote the symmetric system associated to Δ by $\overline{T}$. Then

$$T \to \overline{T}, \ r_{(g,\alpha)} \mapsto \overline{r_{(g,\alpha)}} = r_\alpha$$

is well-defined and provides a surjective system morphism. If $\alpha \in \Delta$ is a root such that $2\alpha \in \Delta$, then $r_\alpha = r_{2\alpha}$. So, in view of (R2) the assignment $r_\alpha \mapsto \dot{r}_\alpha = r_{(0,\alpha)}$ for every $\alpha \in \Delta^{\mathrm{red}}$ defines a map $\overline{T} \to T$, which is a system section for $\overline{\cdot}$.

Let $\mathcal{A}$ be the subgroup of $\mathrm{Aut}_G(G \times \mathcal{L})$ generated by T. Then $\mathcal{A}$ acts on T by conjugation. Denote by $\cdot^{\mathcal{A}} : T \to \mathcal{A}$ the embedding, then $(\mathcal{A}, \cdot^{\mathcal{A}})$ is a proper T-reflection group.

According to Remark 7.3 we may identify $\mathcal{A}$ with the subgroup of $(G \otimes \mathcal{L}^\vee) \rtimes \mathrm{Aut}(\mathcal{L})$ generated by $T^{\mathcal{A}}$. By definition $\mathcal{V}$ is the subgroup of $\mathrm{Aut}(\mathcal{L})$ generated by the reflections r_Δ. Let $\mathcal{K}$ be the subgroup of $\mathrm{Hom}(\mathcal{L}, G)$ generated by

$$\{ g \otimes \alpha^\vee \mid (g, \alpha) \in R \}.$$

Then $\mathcal{A} = \mathcal{K} \rtimes \mathcal{V}$. Moreover the projection $\mathcal{K} \rtimes \mathcal{V} \to \mathcal{V}$ is $\overline{\cdot}$-compatible, and since $\mathcal{V}$ is the initial $\overline{T}$-reflection group, we have $\overline{\mathcal{A}} = \mathcal{V}$. We have

$$t^{\mathcal{K}} = t^{\mathcal{A}} \dot{t}^{\mathcal{A}} = (g \otimes \alpha^\vee, r_\alpha)(0, r_\alpha) = (g \otimes \alpha^\vee, 1) \tag{8}$$

for all $(g, \alpha) \in R$ and $t = r_{(g,\alpha)}$. So $\mathcal{K}$ can be identified with $\mathcal{K}_{\mathcal{V}}^{\mathcal{A}}$ with the notation from (3).

Set $\mathcal{K}_{\mathrm{fix}} := \{ k \in \mathcal{K} \mid v.k = k \text{ for every } v \in \mathcal{V} \}$.

Lemma 7.5 $\mathcal{K}_{\mathrm{fix}} = \{0\}$

Proof. Let $k \in \mathcal{K}_{\mathrm{fix}}$. So $k \in \mathrm{Hom}(\mathcal{L}, G)$ with $k(v.\lambda - \lambda) = 0$ for all $\lambda \in \mathcal{L}$ and $v \in \mathcal{V}$. In other words $k(\mathcal{L}_{\mathrm{eff}}) = \{0\}$. Since $2\mathcal{L} \subseteq \mathcal{L}_{\mathrm{eff}}$ due to Proposition 6.5 and since G is torsion-free, we can conclude $k = 0$. ∎

Lemma 7.6 *If G is non-trivial, then the center of $\mathcal{A}$ is trivial.*

Proof. If $k_1, k_2 \in \mathcal{K}$ and $v \in \mathcal{V}$ then

$$\begin{aligned}
[(k_1, v), (k_2, 1)] &= (k_1, v)(k_2, 1)(-v^{-1}.k_1, v^{-1})(-k_2, 1) \\
&= (k_1 + v.k_2, v)(-v^{-1}.k_1 - v^{-1}.k_2, v^{-1}) \\
&= (k_1 + v_1.k_2 - k_1 - k_2, 1) = (v_1.k_2 - k_2, 1). \tag{9}
\end{aligned}$$

Let $\alpha \in \Delta$. Since G is torsion-free, axiom (R1) and (7) imply that $\langle S_\alpha \rangle$ is not just the trivial subgroup of G. So there is a non-trivial $g \in G$ such that $(g, \alpha) \in R$. Suppose that (k_1, v) is in the center of $\mathcal{A}$. Let $(k_2, 1) = t^{\mathcal{K}} = (g \otimes \alpha^\vee, 1)$ with $t = r_{(g,\alpha)}$. Then (9) yields $g \otimes \alpha^\vee - g \otimes v.\alpha^\vee = g \otimes (\alpha^\vee - v.\alpha^\vee) = 0$. Since G is torsion-free, this implies $v.\alpha^\vee = \alpha^\vee$. Since $\alpha \in \Delta$ was arbitrary, we may conclude $v = 1$. So any element of the center of $\mathcal{A}$ is of the form $(k_1, 1)$.

Now suppose $(k_2, 1)$ is in the center of $\mathcal{A}$. Then (9) implies $k_2 \in \mathcal{K}_{\mathrm{fix}}$. According to Lemma 7.5, we have $k_2 = 0$. ∎

Corollary 7.7 *If G is not trivial, then $\mathcal{A}$ is the terminal T-reflection group.*

Proof. Suppose $\mathcal{B}$ is the terminal T-reflection group. Then there is a reflection morphism $\mathcal{A} \to \mathcal{B}$. This is a central extension by Lemma 2.10. So it is injective by Lemma 7.6. ∎

Remark 7.8 Suppose R is of non-reduced type, i.e. of type $BC_\ell(\ell \geq 1)$. As done in [Hof07], we will associate a trimmed version of R to it. However, here we will accomplish this by turning short roots into extra long ones and not the other way around, as in the cited article. For a root $(g, \alpha) \in R$ we set

$$\mathrm{trim}(g, \alpha) \;=\; \begin{cases} (2g, 2\alpha) & \text{if } \alpha \in \Delta_{\mathrm{sh}}, \\ (g, \alpha) & \text{otherwise.} \end{cases}$$

Then we define

$$R' := \mathrm{trim}(R) \subseteq R^{\mathrm{lg}} \cup R^{\mathrm{ex}}, \quad \Delta' := \Delta^{\mathrm{lg}} \cup \Delta^{\mathrm{ex}} \quad \text{and} \quad G' := \langle S_{\mathrm{lg}} \rangle \subseteq G.$$

Then Δ' is an irreducible finite root system. Its associated symmetric system can be identified with the symmetric system $\overline{T}$ associated to Δ. The Weyl group $\mathcal{V}'$ of Δ' then coincides, as a $\overline{T}$-reflection group, with $\mathcal{V}$. Since we have

$$r_{\mathrm{trim}(\alpha)} = r_\alpha \quad \text{and} \quad r_\alpha.\mathrm{trim}(\beta) = \mathrm{trim}(r_\alpha.\beta)$$

for all α and $\beta \in R$, the set R' is a root system of kind Δ' extended by G'.

Note that we have $2(G \times \mathcal{L}) \subseteq G' \times \mathcal{L}' \subseteq G \times \mathcal{L}$. Since $G \times \mathcal{L}$ is torsion-free, every element $\mathrm{Aut}(G' \times \mathcal{L}')$ extends to a unique element of $\mathrm{Aut}(G \times \mathcal{L})$. This provides an injective group homomorphism $\mathrm{Aut}(G' \times \mathcal{L}') \to \mathrm{Aut}(G \times \mathcal{L})$. This homomorphism maps $r'_{\mathrm{trim}(\alpha)}$ to r_α for every $\alpha \in R$. So it restricts to an isomorphism $\mathcal{A}' \to \mathcal{A}$. Using this isomorphism, we may identify the symmetric systems T' and T.

Since our further investigations only depend on the symmetric system T with its terminal reflection group $\mathcal{A}$, we may restrict ourselves to the case of a reduced root system Δ. ◇

From now on, let Δ be a *reduced* irreducible root system. If R is of simply laced type, i.e. only contains short roots, then $\mathcal{K} = G \otimes \mathcal{L}^\vee$ since S_{sh} generates G. If R is of non-simply laced type, we will use the concept of a twist decomposition to understand $\mathcal{K}$ better.

Definition 7.9 (Twist decomposition) Suppose Δ is of non-simply laced type. A decomposition $G = G_1 \oplus G_2$ as a direct sum is called a *twist decomposition (for R)* if the following conditions are satisfied:

(T1) $S_{\mathrm{sh}} = (S_{\mathrm{sh}} \cap G_1) + \langle S_{\mathrm{lg}} \cap G_2 \rangle$

(T2) $S_{\mathrm{lg}} = k_\Delta \langle S_{\mathrm{sh}} \cap G_1 \rangle + (S_{\mathrm{lg}} \cap G_2)$

A root system R will be called *tame* if it is of non-simply-laced type and G has a twist decomposition, or if it is of simply laced type. ◇

If G is finitely generated and free then R is tame. (See [Aza99] Lemma 1.19). The twist number defined in [Aza99] Definition 1.18 in the case of EARSs is the rank of G_1 for a corresponding twist decomposition $G = G_1 \oplus G_2$.

Remark 7.10 In view of the fact that $\langle S_{\mathrm{sh}} \rangle = G$, a decomposition $G = G_1 \oplus G_2$ is a twist decomposition if and only if the following conditions are satisfied:

(i) $S_{\mathrm{sh}} = (S_{\mathrm{sh}} \cap G_1) + G_2$,

(ii) $S_{\mathrm{lg}} = k_\Delta G_1 + (S_{\mathrm{lg}} \cap G_2)$,

(iii) $\langle G_2 \cap S_{\mathrm{lg}} \rangle = G_2$.

If it is a twist decomposition, then

(iv) $\langle G_1 \cap S_{\mathrm{sh}} \rangle = G_1$,

(v) $\langle S_{\mathrm{sh}} \rangle = G_1 + G_2$ and $\langle S_{\mathrm{lg}} \rangle = k_\Delta G_1 + G_2$. ◇

From now on suppose that R is tame. If we have a twist decomposition $G = G_1 \oplus G_2$ then we will use the identification of $\mathcal{L} \subseteq \mathcal{L}^\vee$ from Lemma 6.3. In this case we have

Lemma 7.11 $\mathcal{K} = (G_1 \otimes \mathcal{L}) \oplus (G_2 \otimes \mathcal{L}^\vee)$.

Proof. Recall (8) and the fact that $\mathcal{K}$ is the subgroup of

$$G \otimes \mathcal{L}^\vee = (G_1 \otimes \mathcal{L}^\vee) \oplus (G_2 \otimes \mathcal{L}^\vee)$$

generated by $T^{\mathcal{K}}$. Let $T^{\mathcal{K}}_{\mathrm{sh}}$ be the subset of $\mathcal{K}$ consisting of the elements $r^{\mathcal{K}}_\alpha$ with $\alpha \in \Delta_{\mathrm{sh}}$ and let $T^{\mathcal{K}}_{\mathrm{lg}}$ be the subset of $\mathcal{K}$ consisting of the elements $r^{\mathcal{K}}_\alpha$ with $\alpha \in \Delta_{\mathrm{lg}}$. Then

$$\langle T^{\mathcal{K}}_{\mathrm{sh}} \rangle = (G_1 \otimes \mathcal{L}) + (G_2 \otimes \mathcal{L}) \quad \text{and} \quad \langle T^{\mathcal{K}}_{\mathrm{lg}} \rangle = (k_\Delta G_1 \otimes \mathcal{L}^\vee) + (G_2 \otimes \mathcal{L}^\vee).$$

Since $k\mathcal{L}^\vee \subseteq \mathcal{L} \subseteq \mathcal{L}^\vee$ by Lemma 6.3 we obtain the claim of the Lemma. ∎

Definition 7.12 A reflection bihomomorphism $b : \mathcal{K}^2 \to \mathcal{Z}$ is called *admissible* if the following condition is satisfied: If s and $t \in T$ with $\overline{s} \perp \overline{t}$ then $b\big(s^{\mathcal{K}}, t^{\mathcal{K}}\big) = 0$. ◇

Remark 7.13 There is an abelian group $\mathcal{K} \curlywedge \mathcal{K}$ and a admissible reflection bihomomorphism

$$\curlywedge : \mathcal{K}^2 \to \mathcal{K} \curlywedge \mathcal{K}, \ (x, y) \mapsto x \curlywedge y$$

such that the following universal property is satisfied: For any admissible $b :$ $\mathcal{K}^2 \to B$ there is a unique group homomorphism $\varphi : \mathcal{K} \curlywedge \mathcal{K} \to B$ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{K}^2 & \longrightarrow & \mathcal{K} \curlywedge \mathcal{K} \\
 & \searrow{\scriptstyle b} & \downarrow{\scriptstyle \varphi} \\
 & & B.
\end{array}
$$

The group $\mathcal{K} \curlywedge \mathcal{K}$ can be constructed as the quotient of $\mathcal{K} \otimes \mathcal{K}$ by the subgroup generated by the union of the sets

$$
\begin{aligned}
M_1 &= \{t_1^{\mathcal{K}} \otimes t_2^{\mathcal{K}} \mid t_1 \text{ and } t_2 \in T \text{ with } t_1 \perp t_2\}, \\
M_2 &= \{t_1^{\mathcal{K}} \otimes t_2^{\mathcal{K}} - s^{\mathcal{V}}.t_1^{\mathcal{K}} \otimes s^{\mathcal{V}}.t_2^{\mathcal{K}} \mid t_1, t_2 \in T \text{ and } s \in S\}, \\
M_3 &= \{t^{\mathcal{K}} \otimes t^{\mathcal{K}} \mid t \in T\}, \\
M_4 &= \{t_1^{\mathcal{K}} \otimes t_2^{\mathcal{K}} + t_2^{\mathcal{K}} \otimes t_1^{\mathcal{K}} \mid t_1 \text{ and } t_2 \in T\} \quad \text{and} \\
M_5 &= \{t^{\mathcal{K}} \otimes s^{\mathcal{V}}.t^{\mathcal{K}} \mid t \in T \text{ and } s \in S\}.
\end{aligned}
$$

$\diamond$

Recall that R is supposed to be tame and consider the map

$$(G \otimes \mathcal{L}^{\vee})^2 \to (G \wedge G) \otimes (\mathcal{L}^{\vee} \boxtimes \mathcal{L}^{\vee}), \ (g, \lambda, h, \mu) \mapsto (g \wedge h) \otimes (\lambda \boxtimes \mu).$$

Its restriction to $\mathcal{K}^2$ is admissible. So, by the universal property of $\mathcal{K} \curlywedge \mathcal{K}$ there is a group homomorphism ψ such that the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{K}^2 & \longrightarrow & \mathcal{K} \curlywedge \mathcal{K} \\
 & \searrow & \downarrow{\scriptstyle \psi} \\
 & & (G \wedge G) \otimes (\mathcal{L}^{\vee} \boxtimes \mathcal{L}^{\vee}).
\end{array}
$$

Theorem 7.14 *The map ψ is injective and if R is simply laced, it is surjective.*

Proof. Throughout this proof we will take advantage of the fact that the tensor product is commutative and associative and identify $\mathcal{K} \otimes \mathcal{K}$ with the subgroup of $G \otimes G \otimes \mathcal{L}^{\vee} \otimes \mathcal{L}^{\vee}$ generated by $\{g \otimes h \otimes \alpha \otimes \beta \mid (g, \alpha) \text{ and } (h, \beta) \in R\}$.

First we assume that R is simply laced, so $\mathcal{K} \cong G \otimes \mathcal{L}^{\vee}$. We will use the sets M_1 to M_5 defined in Remark 7.13 and restate them here. Let $\mathcal{K} \boxtimes \mathcal{K}$ be the quotient of $\mathcal{K} \otimes \mathcal{K}$ by the subgroup generated by the union of the sets

$$M_1 = \{g \otimes h \otimes \alpha^{\vee} \otimes \beta^{\vee} \mid (g, \alpha), (h, \beta) \in R \text{ with } r_{\alpha} \perp r_{\beta}\} \quad \text{and}$$
$$M_2 = \{g \otimes h \otimes (\alpha^{\vee} \otimes \beta^{\vee} - (r_{\gamma}.\alpha^{\vee}) \otimes (r_{\gamma}.\beta^{\vee})) \mid (g, \alpha), (h, \beta) \in R \text{ and } \gamma \in \Delta\}.$$

By $\pi : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K} \boxtimes \mathcal{K}$ we denote the quotient morphism. Recall the properties of $\mathcal{L}^{\vee} \boxtimes \mathcal{L}^{\vee}$ given in Corollary 6.10. We have

$$\mathcal{K} \boxtimes \mathcal{K} \cong G \otimes G \otimes (\mathcal{L}^{\vee} \boxtimes \mathcal{L}^{\vee}).$$

Now set

$$
\begin{aligned}
M_3 &= \{g \otimes g \otimes \alpha^\vee \otimes \alpha^\vee \mid (g,\alpha) \in R\}, \\
M_4 &= \{g \otimes h \otimes \alpha^\vee \otimes \beta^\vee - h \otimes g \otimes \beta^\vee \otimes \mathcal{V}\alpha^\vee \mid (g,\alpha),(h,\beta) \in R\} \\
M_4' &= \{(g \otimes h - h \otimes g) \otimes \alpha^\vee \otimes \beta^\vee \mid (g,\alpha),(h,\beta) \in R\} \quad \text{and} \\
M_5 &= \{g \otimes g \otimes \alpha^\vee \otimes (r_\gamma.\alpha^\vee) \mid (g,\alpha) \in R \text{ and } \gamma \in \Delta\}.
\end{aligned}
$$

Note that $\pi(M_4) = \pi(M_4')$ since $(\mathcal{L}^\vee)^2 \to \mathcal{L}^\vee \boxtimes \mathcal{L}^\vee$ is symmetric according to Theorem 6.7. So $\mathcal{K} \curlywedge \mathcal{K}$ is the quotient of $\mathcal{K} \boxtimes \mathcal{K}$ by its subgroup generated by $\pi(M_3) \cup \pi(M_4') \cup \pi(M_5)$. Since $\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee$ is generated by

$$
\{\alpha^\vee \boxtimes \alpha^\vee \mid \alpha \in \Delta\} \cup \{\alpha^\vee \boxtimes (r_\beta.\alpha^\vee) \mid \alpha \text{ and } \beta \in \Delta\}
$$

by Corollary 6.10, we have

$$
\mathcal{K} \curlywedge \mathcal{K} \cong (G \wedge G) \otimes (\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee).
$$

Now we assume that R is not simply laced and that there is a twist decomposition $G = G_1 \oplus G_2$. We have

$$
\begin{aligned}
\mathcal{K} \otimes \mathcal{K} &= \big((G_1 \otimes \mathcal{L}) \oplus (G_2 \otimes \mathcal{L}^\vee)\big) \otimes \big((G_1 \otimes \mathcal{L}) \oplus (G_2 \otimes \mathcal{L}^\vee)\big) \\
&= \quad (G_1 \otimes G_1 \otimes \mathcal{L} \otimes \mathcal{L}) \quad \oplus \quad (G_1 \otimes G_2 \otimes \mathcal{L} \otimes \mathcal{L}^\vee) \\
&\quad \oplus \ (G_2 \otimes G_1 \otimes \mathcal{L}^\vee \otimes \mathcal{L}) \quad \oplus \quad (G_2 \otimes G_2 \otimes \mathcal{L}^\vee \otimes \mathcal{L}^\vee).
\end{aligned}
$$

Set

$$
\begin{aligned}
\mathcal{K} \overline{\boxtimes} \mathcal{K} &= \quad \big(G_1 \otimes G_1 \otimes (\mathcal{L} \boxtimes \mathcal{L})\big) \quad \oplus \quad \big(G_1 \otimes G_2 \otimes (\mathcal{L} \boxtimes \mathcal{L}^\vee)\big) \\
&\quad \oplus \ \big(G_2 \otimes G_1 \otimes (\mathcal{L}^\vee \boxtimes \mathcal{L})\big) \quad \oplus \quad \big(G_2 \otimes G_2 \otimes (\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee)\big).
\end{aligned}
$$

Then $\mathcal{K} \overline{\boxtimes} \mathcal{K}$ can be obtained as a quotient of $\mathcal{K} \otimes \mathcal{K}$ by some subgroup of the group $\langle M_1 \cup M_2 \rangle$. Denote the quotient homomorphism by $\overline{\pi} : \ \mathcal{K} \otimes \mathcal{K} \to \mathcal{K} \overline{\boxtimes} \mathcal{K}$.

Now set

$$
\begin{aligned}
\mathcal{K} \overline{\curlywedge} \mathcal{K} &= \quad \big((G_1 \wedge G_1) \otimes (\mathcal{L} \boxtimes \mathcal{L})\big) \quad \oplus \quad \big((G_1 \otimes G_2) \otimes (\mathcal{L} \boxtimes \mathcal{L}^\vee)\big) \\
&\quad \oplus \ \big((G_2 \wedge G_2) \otimes (\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee)\big).
\end{aligned}
$$

Then $\mathcal{K} \overline{\curlywedge} \mathcal{K}$ can be obtained as a quotient of $\mathcal{K} \overline{\boxtimes} \mathcal{K}$ by some subgroup of the group $\langle \overline{\pi}(M_3 \cup M_4 \cup M_4' \cup M_5) \rangle$. For this, note that $\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee$ is generated by

$$
\{\alpha^\vee \boxtimes \alpha^\vee \mid \alpha \in \Delta_{\mathrm{lg}}\} \cup \{\alpha^\vee \boxtimes (r_\beta.\alpha^\vee) \mid \alpha \in \Delta_{\mathrm{lg}}, \beta \in \Delta\}
$$

and with the identification $\mathcal{L} \subseteq \mathcal{L}^\vee$ from Lemma 6.3 the group $\mathcal{L} \boxtimes \mathcal{L}$ is generated by $\{\alpha^\vee \otimes_\mathcal{V} \alpha^\vee \mid \alpha \in \Delta_{\mathrm{sh}}\} \cup \{\alpha^\vee \otimes_\mathcal{V} (r_\beta.\alpha^\vee) \mid \alpha \in \Delta_{\mathrm{sh}}, \beta \in \Delta\}$.

By Corollary 6.16, we have $\mathcal{L} \boxtimes \mathcal{L} \subseteq \mathcal{L} \boxtimes \mathcal{L}^\vee \subseteq \mathcal{L}^\vee \boxtimes \mathcal{L}^\vee$. So $\mathcal{K} \overline{\boxtimes} \mathcal{K}$ is a subgroup of

$$
\begin{aligned}
G \wedge G \otimes (\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee) &= \quad \big((G_1 \wedge G_1) \otimes (\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee)\big) \\
&\quad \oplus \ \big((G_1 \otimes G_2) \otimes (\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee)\big) \\
&\quad \oplus \ \big((G_2 \wedge G_2) \otimes (\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee)\big).
\end{aligned}
$$

By the way we have constructed $\mathcal{K}\overline{\curlywedge}\mathcal{K}$, there is a surjective homomorphism $\mathcal{K}\overline{\curlywedge}\mathcal{K} \to \mathcal{K} \curlywedge \mathcal{K}$ and overall we have the following commuting diagram:

$$
\begin{array}{ccc}
 & & \mathcal{K}\overline{\curlywedge}\mathcal{K} \\
 & \nearrow & \downarrow \\
\mathcal{K}^2 & \longrightarrow & \mathcal{K} \curlywedge \mathcal{K} \quad \varphi \\
 & \searrow & \downarrow \psi \\
 & & (G \wedge G) \otimes (\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee),
\end{array}
$$

where φ is injective. This means that the map ψ is injective. ∎

Corollary 7.15 *The group $\mathcal{K} \curlywedge \mathcal{K}$ is isomorphic to a subgroup of $G \wedge G$ and is thus torsion-free.*

Proof. By the preceding theorem, the group $\mathcal{K} \curlywedge \mathcal{K}$ is isomorphic to a subgroup of $(G \wedge G) \otimes (\mathcal{L}^\vee \boxtimes \mathcal{L}^\vee)$, which in turn is isomorphic to $G \wedge G$ by Corollary 6.10. ∎

Definition 7.16 (Weyl group) Suppose R is tame. Then the T-reflection group $\mathcal{W}(\mathcal{A}, \curlywedge)$ is called the *Weyl group* of R. ◇

This new notion of a Weyl group coincides with the existing notion of a Weyl group when R can be interpreted as an EARS as in Remark 7.2.

Recall that $c_\mathcal{U} : \mathcal{K}^2 \to (\mathcal{K}^\mathcal{U})'$ (defined in Remark 5.3) is a reflection bihomomorphism due to Proposition 5.6.

Lemma 7.17 *The reflection bihomomorphism $c_\mathcal{U}$ is admissible.*

Proof. Suppose s and $t \in T$. There are (g, α) and $(h, \beta) \in R$ such that $s = r_{(g,\alpha)}$ and $t = r_{(h,\beta)}$. Now we suppose $\overline{s} \perp \overline{t}$, in other words $r_\alpha \perp r_\beta$. This implies $\langle \alpha^\vee, \beta \rangle = 0$. So we have

$$
s.t = r_{(g,\alpha)}.r_{(h,\beta)} = r_{(h + \langle \alpha^\vee, \beta \rangle g, r_\alpha.\beta)} = r_{(h,\beta)} = t.
$$

This entails

$$
[s^\mathcal{U}, t^\mathcal{U}] = [s^\mathcal{U}, \dot{t}^\mathcal{U}] = [\dot{s}^\mathcal{U}, t^\mathcal{U}] = [\dot{s}^\mathcal{U}, \dot{t}^\mathcal{U}] = 1 \quad \text{and thus}
$$

$$
\left[s^\mathcal{U}\dot{s}^\mathcal{U}, t^\mathcal{U}\dot{t}^\mathcal{U}\right] = s^\mathcal{U}\dot{s}^\mathcal{U}t^\mathcal{U}\dot{t}^\mathcal{U}\dot{s}^\mathcal{U}s^\mathcal{U}\dot{t}^\mathcal{U}t^\mathcal{U} = t^\mathcal{U}\dot{t}^\mathcal{U}s^\mathcal{U}\dot{s}^\mathcal{U}\dot{s}^\mathcal{U}s^\mathcal{U}\dot{t}^\mathcal{U}t^\mathcal{U} = 1
$$

in $\mathcal{U}$. Written additively, in the center of $\mathcal{U}$ this means $c_\mathcal{U}\left(s^\mathcal{K}, t^\mathcal{K}\right) = 0$. ∎

The T-reflection morphism $\mathcal{U} \to \mathcal{W}$ restricts to a surjective group homomorphism $\varphi : (\mathcal{K}^\mathcal{U})' \to (\mathcal{K}^\mathcal{W})'$.

Proposition 7.18 *The map φ is an isomorphism.*

Proof. The way $c_{\mathcal{U}}$ and $c_{\mathcal{W}}$ are defined in Remark 5.3 the following diagram commutes:

$$
\begin{array}{ccc}
 & & \mathcal{K} \curlywedge \mathcal{K} \\
 & \nearrow{\scriptstyle \curlywedge} & \downarrow \\
\mathcal{K}^2 \xrightarrow{\ c_{\mathcal{U}}\ } & (\mathcal{K}^{\mathcal{U}})' & \\
 & \searrow{\scriptstyle c_{\mathcal{W}}} & \downarrow{\varphi} \quad \text{id} \\
 & \searrow{\scriptstyle \curlywedge} & (\mathcal{K}^{\mathcal{W}})' \\
 & & \downarrow \\
 & & \mathcal{K} \curlywedge \mathcal{K}.
\end{array}
$$

The map $(\mathcal{K}^{\mathcal{W}})' \to \mathcal{K} \curlywedge \mathcal{K}$ exists due to Corollary 5.5. The map $\mathcal{K} \curlywedge \mathcal{K} \to (\mathcal{K}^{\mathcal{U}})'$ exists by the universal property of $\curlywedge$. It is surjective, so φ is injective. ∎

Due to the preceding proposition we have the commutative diagram

$$
\begin{array}{ccccc}
1 & \longrightarrow & (\mathcal{K}^{\mathcal{U}})' & \longrightarrow & (\mathcal{K}^{\mathcal{W}})' \\
\downarrow & & \downarrow & & \downarrow \\
\ker(\mathcal{U} \to \mathcal{W}) & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{W} \\
\downarrow{\psi} & & \downarrow & & \downarrow \\
\ker(\mathcal{U}_{\mathrm{ab}} \to \mathcal{W}_{\mathrm{ab}}) & \longrightarrow & \mathcal{U}_{\mathrm{ab}} & \longrightarrow & \mathcal{W}_{\mathrm{ab}}
\end{array}
$$

and by Lemma 3.15 we obtain

Corollary 7.19 *The map ψ is an isomorphism.*

We are no ready to prove the main result of this article. Let Δ be a reduced finite irreducible root system. Let G be a free abelian group. Let R be a tame root system of kind Δ extended by G. Let T be the symmetric system associated to R. Let $\mathcal{U}$ be the initial T-reflection group and let $\mathcal{W}$ be the Weyl group. Using the ab-functor we obtain the following commutative diagram:

$$
\begin{array}{ccccc}
\ker(\mathcal{U} \to \mathcal{W}) & \longrightarrow & \mathcal{U} & \longrightarrow & \mathcal{W} \\
\downarrow & & \downarrow & & \downarrow \\
\ker(\mathcal{U}^{\mathrm{ab}} \to \mathcal{W}^{\mathrm{ab}}) & \longrightarrow & \mathcal{U}^{\mathrm{ab}} & \longrightarrow & \mathcal{W}^{\mathrm{ab}},
\end{array}
$$

where $\mathcal{U} \to \mathcal{W}$ is a T-reflection morphism, $\mathcal{U}^{\mathrm{ab}} \to \mathcal{W}^{\mathrm{ab}}$ is a T^{ab}-reflection morphism and $\mathcal{U} \to \mathcal{U}^{\mathrm{ab}}$ and $\mathcal{W} \to \mathcal{W}^{\mathrm{ab}}$ are ab-compatible. The groups $\mathcal{U}^{\mathrm{ab}}$ and $\mathcal{W}^{\mathrm{ab}}$ are just the abelianizations of $\mathcal{U}$ and $\mathcal{W}$ respectively.

Theorem 7.20 *The map* $\ker(\mathcal{U} \to \mathcal{W}) \to \ker(\mathcal{U}^{\mathrm{ab}} \to \mathcal{W}^{\mathrm{ab}})$ *is an isomorphism.*

Proof. The composition $\ker(\mathcal{U} \to \mathcal{W}) \to \ker(\mathcal{U}_{\mathrm{ab}} \to \mathcal{W}_{\mathrm{ab}}) \to \ker(\mathcal{U}^{\mathrm{ab}} \to \mathcal{W}^{\mathrm{ab}})$ is an isomorphism due to Corollary 7.19 and Proposition 4.2.   ∎

Corollary 7.21 *If* $\ker(\mathcal{U}_{\mathrm{ab}} \to \mathcal{W}_{\mathrm{ab}}) \neq \{0\}$*, then* $\mathcal{U}$ *and* $\mathcal{W}$ *are not isomorphic (as groups).*

Proof.   Suppose $\ker(\mathcal{U}_{\mathrm{ab}} \to \mathcal{W}_{\mathrm{ab}}) \neq \{0\}$. According to Lemma 2.14 and Corollary 7.7 the center of $\mathcal{W}$ is isomorphic to a the kernel of $\mathcal{W} \to \mathcal{A}$, which is isomorphic to a subgroup of $\mathcal{K} \curlywedge \mathcal{K}$ and is thus torsion-free by Corollary 7.15. The center of $\mathcal{U}$ contains $\ker(\mathcal{U} \to \mathcal{W})$, which contains elements of order two. ∎

Remark 7.22 Since $\mathcal{A} = \mathcal{K} \rtimes \mathcal{V}$, the commutator subgroup of $\mathcal{A}$ is given by

$$\mathcal{A}' = \mathcal{K}_{\mathrm{eff}} \rtimes \mathcal{V}' \ \text{ with } \ \mathcal{K}_{\mathrm{eff}} := [\mathcal{A}, \mathcal{K}] = \langle k - v.k \mid k \in \mathcal{K}, v \in \mathcal{V} \rangle.$$

We have

$$\mathcal{K}_{\mathrm{eff}} = G_1 \otimes \mathcal{L}_{\mathrm{eff}} \oplus G_2 \otimes \mathcal{L}_{\mathrm{eff}}^{\vee}.$$

We set $^{\mathrm{ab}}\mathcal{K} := \mathcal{K}/\mathcal{K}_{\mathrm{eff}}$ and thus have $\mathcal{A}^{\mathrm{ab}} = {}^{\mathrm{ab}}\mathcal{K} \times \mathcal{V}^{\mathrm{ab}}$. The group $\mathcal{V}^{\mathrm{ab}}$ is the initial $\overline{T}^{\mathrm{ab}}$-reflection group and thus it is a proper reflection group according to Construction 3.11.

   If Δ is simply laced, then Proposition 6.5 implies

$$^{\mathrm{ab}}\mathcal{K} \cong G \otimes \frac{\mathcal{L}^{\vee}}{\mathcal{L}_{\mathrm{eff}}^{\vee}} \cong \begin{cases} G \otimes \mathbb{Z}_2 & \text{if } R \text{ is of type } A_1, \\ \{0\} & \text{otherwise.} \end{cases}$$

If Δ is non-simply laced and $G = G_1 \oplus G_2$ is a twist decomposition, then $^{\mathrm{ab}}\mathcal{K}$ is isomorphic to

$$^{\mathrm{ab}}\mathcal{K} \cong \left(G_1 \otimes \frac{\mathcal{L}}{\mathcal{L}_{\mathrm{eff}}} \right) \oplus \left(G_2 \otimes \frac{\mathcal{L}^{\vee}}{\mathcal{L}_{\mathrm{eff}}^{\vee}} \right) \cong \begin{cases} G \otimes \mathbb{Z}_2 & \text{if } R \text{ is of type } B_2, \\ G_1 \otimes \mathbb{Z}_2 & \text{if } R \text{ is of type } B_\ell(\ell \geq 3), \\ G_2 \otimes \mathbb{Z}_2 & \text{if } R \text{ is of type } C_\ell(\ell \geq 3), \\ 0 & \text{if } R \text{ is of type } F_4 \text{ or } G_2. \end{cases} \diamond$$

Proposition 7.23 *We have*

$$\mathcal{A}.(h, \beta) = \begin{cases} (h + \langle \Delta^{\vee}, \beta \rangle \langle S_{\mathrm{sh}} \rangle, \mathcal{V}.\beta) & \text{in the simply laced case,} \\ (h + \langle \Delta_{\mathrm{lg}}^{\vee}, \beta \rangle \langle S_{\mathrm{sh}} \rangle + \langle \Delta_{\mathrm{sh}}^{\vee}, \beta \rangle \langle S_{\mathrm{lg}} \rangle, \mathcal{V}.\beta), & \text{otherwise.} \end{cases}$$

Type	$\langle \Delta_{\mathrm{sh}}^{\vee}, \Delta_{\mathrm{sh}} \rangle$	$\langle \Delta_{\mathrm{sh}}^{\vee}, \Delta_{\mathrm{lg}} \rangle$	$\langle \Delta_{\mathrm{lg}}^{\vee}, \Delta_{\mathrm{sh}} \rangle$	$\langle \Delta_{\mathrm{lg}}^{\vee}, \Delta_{\mathrm{lg}} \rangle$
A_1	$\{\pm 2\}$	$\varnothing$	$\varnothing$	$\varnothing$
B_2	$\{\pm 1\}$	$\{0, \pm 2\}$	$\{0, \pm 2\}$	$\{\pm 2\}$
$B_\ell(\ell \geq 3)$	$\{0, \pm 1\}$	$\{0, \pm 1, \pm 2\}$	$\{0, \pm 2\}$	$\{0, \pm 2\}$
$C_\ell(\ell \geq 3)$	$\{0, \pm 1\}$	$\{0, \pm 2\}$	$\{0, \pm 1, \pm 2\}$	$\{0, \pm 2\}$
F_4	$\{0, \pm 1\}$	$\{0, \pm 1, \pm 2\}$	$\{0, \pm 1, \pm 2\}$	$\{0, \pm 2\}$
G_2	$\{0, \pm 1\}$	$\{0, \pm 1, \pm 2\}$	$\{0, \pm 1, \pm 2\}$	$\{0, \pm 3\}$
other	$\{0, \pm 1, \pm 2\}$	$\varnothing$	$\varnothing$	$\varnothing$

Table 2: Pairing values for the finite reduced irreducible root systems

More precisely,

$$
\mathcal{A}.(h, \beta) = \begin{cases}
(h + 2G, \Delta) & \text{if } R \text{ is of type } A_1, \\
(h + 2G_1 + G_2, \Delta_{\mathrm{sh}}) & \text{if } R \text{ is of type } B_2 \text{ and } \beta \in \Delta_{\mathrm{sh}}, \\
(h + 2G_1 + 2G_2, \Delta_{\mathrm{lg}}) & \text{if } R \text{ is of type } B_2 \text{ and } \beta \in \Delta_{\mathrm{lg}}, \\
(h + 2G_1 + G_2, \Delta_{\mathrm{sh}}) & \text{if } R \text{ is of type } B_\ell(\ell \geq 3) \text{ and } \beta \in \Delta_{\mathrm{sh}}, \\
(h + 2G_1 + G_2, \Delta_{\mathrm{lg}}) & \text{if } R \text{ is of type } B_\ell(\ell \geq 3) \text{ and } \beta \in \Delta_{\mathrm{lg}}, \\
(h + G_1 + G_2, \Delta_{\mathrm{sh}}) & \text{if } R \text{ is of type } C_\ell(\ell \geq 3) \text{ and } \beta \in \Delta_{\mathrm{sh}}, \\
(h + 2G_1 + 2G_2, \Delta_{\mathrm{lg}}) & \text{if } R \text{ is of type } C_\ell(\ell \geq 3) \text{ and } \beta \in \Delta_{\mathrm{lg}}, \\
(G, \Delta) & \text{if } R \text{ is of any other type.}
\end{cases}
$$

Proof. The first equation in the proposition can be derived in the same way as done in the proof of [Hof07] Proposition 5.1. What follows can be derived from it using $\langle S_{\mathrm{sh}} \rangle = G_1 + G_2$ and $\langle S_{\mathrm{lg}} \rangle = k_\Delta G_1 + G_2$ and Table 2, the data of which is taken from [Aza99] with one correction. ∎

Remark 7.24 Under the hypothesis of Theorem 7.20 the word problem for the presentation (1) of $\mathcal{U}$ has the following solution: The element $t_1^{\mathcal{U}} t_2^{\mathcal{U}} \cdots t_n^{\mathcal{U}}$ is trivial in $\mathcal{U}$ if and only if the following conditions are satisfied:

(i) The element $\overline{t_1}^{\mathcal{W}} \overline{t_2}^{\mathcal{W}} \cdots \overline{t_n}^{\mathcal{W}}$ is trivial in $\mathcal{V}$.

(ii) The element $t_1^{\mathcal{K}^{\mathcal{W}}} t_2^{\mathcal{K}^{\mathcal{W}}} \cdots t_n^{\mathcal{K}^{\mathcal{W}}}$ is trivial in $\mathcal{K}^{\mathcal{W}}$.

(iii) The element $(t_1^{\mathrm{ab}})^{\mathcal{U}^{\mathrm{ab}}} (t_2^{\mathrm{ab}})^{\mathcal{U}^{\mathrm{ab}}} \cdots (t_n^{\mathrm{ab}})^{\mathcal{U}^{\mathrm{ab}}}$ is trivial in $\mathcal{U}^{\mathrm{ab}}$.

Condition (i) can be decided using the solution for the word problem for Coxeter groups (see for instance,[Hum92]). Condition (ii) can be decided by evaluating the elements in $\mathcal{K}^{\mathcal{W}}$. In view of Construction 3.11, condition (iii) can be decided using Proposition 7.23. ◇

Theorem 7.25 *The T^{ab}-reflection group $\mathcal{A}^{\mathrm{ab}}$ is proper.*

Proof. Recall that $\mathcal{A}^{\mathrm{ab}} \cong {}^{\mathrm{ab}}\mathcal{K} \times \mathcal{V}^{\mathrm{ab}}$. The map $T^{\mathrm{ab}} \to \mathcal{A}^{\mathrm{ab}}$ does not contain 0 in its image, since the map $\overline{T}^{\mathrm{ab}} \to \mathcal{V}^{\mathrm{ab}}$ does not. In the following we will show that the map $T^{\mathrm{ab}} \to \mathcal{A}^{\mathrm{ab}}$ is injective.

Let (h_1, β_1) and $(h_2, \beta_2) \in R$. Set $t_1 := r_{(h_1,\beta_1)}$ and $t_2 := r_{(h_2,\beta_2)}$. Now suppose $(t_1^{\mathrm{ab}})^{\mathcal{A}^{\mathrm{ab}}} = (t_2^{\mathrm{ab}})^{\mathcal{A}^{\mathrm{ab}}}$. So we have

$$(h_1 \otimes \beta_1^\vee, r_{\beta_1})^{\mathrm{ab}} = (t_1^{\mathcal{A}})^{\mathrm{ab}} = (t_1^{\mathrm{ab}})^{\mathcal{A}^{\mathrm{ab}}} = (t_2^{\mathrm{ab}})^{\mathcal{A}^{\mathrm{ab}}} = (t_2^{\mathcal{A}})^{\mathrm{ab}} = (h_2 \otimes \beta_2^\vee, r_{\beta_2})^{\mathrm{ab}}.$$

So β_1 and β_2 are in the same $\mathcal{V}$-orbit, say $\beta_1 = v.\beta_2$. Moreover we have

$$h_1 \otimes \beta_1^\vee \equiv h_2 \otimes \beta_2^\vee \equiv h_2 \otimes (v.\beta_2^\vee) \equiv h_2 \otimes \beta_1^\vee \quad \mathrm{mod}\ \mathcal{K}_{\mathrm{eff}},$$

in other words $(h_2 - h_1) \otimes \beta_1^\vee$ is mapped to zero by $\mathcal{K} \to {}^{\mathrm{ab}}\mathcal{K}$. We will distinguish between a number of cases according to the types of R. In each case we will show that (h_1, β_1) and (h_2, β_2) are in the same $\mathcal{A}$-orbit according to Proposition 7.23. This entails $t_1^{\mathrm{ab}} = t_2^{\mathrm{ab}}$. Set $g = h_2 - h_1$.

Case A_1:   In this case we have $g \in 2G$ by Remark 7.22.

For the next three cases keep in mind that according to the twist decomposition, $g \in k_\Delta G_1 + G_2$ if β_1 is long. I those cases $k_\Delta = 2$.

Case B_2:   If $\beta \in \Delta_{\mathrm{sh}}$ then $g \in 2G_1 + G_2$ by Remark 7.22. If $\beta \in \Delta_{\mathrm{lg}}$ then $g \in G_1 + 2G_2$ by Remark 7.22 and thus $g \in 2G_1 + 2G_2$.

Case $B_\ell (\ell \geq 3)$:   If $\beta \in \Delta_{\mathrm{sh}}$ then $g \in 2G_1 + G_2$ by Remark 7.22. If $\beta \in \Delta_{\mathrm{lg}}$ then $g \in 2G_1 + G_2$.

Case $C_\ell (\ell \geq 3)$:   If $\beta \in \Delta_{\mathrm{lg}}$ then $g \in G_1 + 2G_2$ by Remark 7.22 and thus $g \in 2G_1 + 2G_2$.

In the remaining cases there is only a single $\mathcal{A}$-orbit in R, so we are done. ∎

Under the hypotheses of Theorem 7.20 we have

Corollary 7.26 *If Δ is of type $A_\ell(\ell \geq 2)$, $D_\ell(\ell \geq 4)$, $E_\ell(8 \geq \ell \geq 6)$, F_4, or G_2, then $\mathcal{U} \to \mathcal{W}$ is an isomorphism.*

Proof.   In the cases above the symmetric system T^{ab} is a singleton due to Proposition 7.23. Since $\mathcal{A}^{\mathrm{ab}}$ is proper, it is of order two. We have the T^{ab}-reflection morphisms $\mathcal{U}^{\mathrm{ab}} \to \mathcal{W}^{\mathrm{ab}} \to \mathcal{A}^{\mathrm{ab}}$. This forces $\mathcal{U}^{\mathrm{ab}} \to \mathcal{A}^{\mathrm{ab}}$ to be an isomorphism. We are done by Theorem 7.20 ∎

This result generalizes a result about Weyl groups of EARSs of simply laced type, proved for the first time in [Kry00]. It seems very likely that other generalizations of this kind will be obtained by investigating the sequence $\mathcal{U}^{\mathrm{ab}} \to \mathcal{W}^{\mathrm{ab}} \to \mathcal{A}^{\mathrm{ab}}$ of elementary abelian two-groups. Since these are vector spaces over the Galois field with two elements, they are subject to arguments from linear Algebra.

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